Counting pattern-avoiding integer partitions

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Towson University

Based on joint work with
Jonathan Bloom
Lafayette College

PAlmetto **N**umber **T**heory **S**eries
Clemson University
December 14th, 2019

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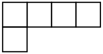
$5 =$ $\quad 4+1 \quad =$ 

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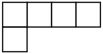
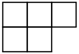
$5 =$ $\quad$ $4+1 \quad =$  $\quad$ $3+2 \quad =$ 

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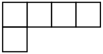
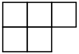
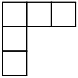
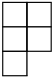
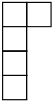
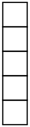
$4+1 =$ 

$3+2 =$ 

$3+1+1 =$ 

$2+2+1 =$ 

$2+1+1+1 =$ 

$1+1+1+1+1 =$ 

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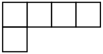
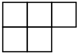
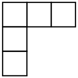
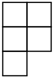
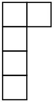
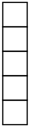
$4+1 =$ 

$3+2 =$ 

$3+1+1 =$ 

$2+2+1 =$ 

$2+1+1+1 =$ 

$1+1+1+1+1 =$ 

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Such configuration are called **Ferrers boards**.

Partition Patterns

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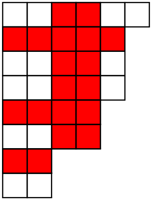
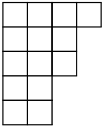
$\alpha =$ 

$\mu =$  .

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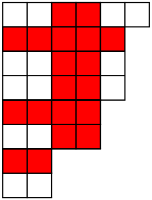
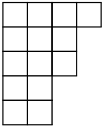
$\alpha =$  $\mu =$  .

We will refer to a fixed partition μ as a **pattern**.

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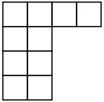
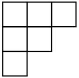
$\alpha =$  $\mu =$  .

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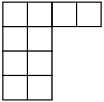
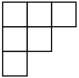
$\alpha =$  $\mu =$  .

We define $\mathrm{Av}_n(\mu)$ to be the set of all μ-avoiding partitions of $n \geq 0$ and set

$$\mathrm{Av}(\mu) = \bigcup_{n \geq 0} \mathrm{Av}_n(\mu).$$

Sequences

Motivating Question: For a fixed pattern μ, what can we say about the sequence

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as well as the asymptotic growth rate of $|\mathrm{Av}_n(\mu)|$.

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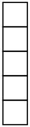

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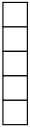

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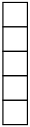

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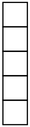

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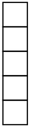

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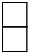
- $\mu = (1, 1) =$ . Any partition λ at least 2 parts contains μ. The only partition of size n avoiding μ is $n = n$.

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  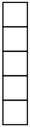

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  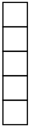

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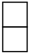
- $\mu = (1,1) =$ . Any partition λ at least 2 parts contains μ. The only partition of size n avoiding μ is $n = n$.

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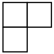
$\mu = (2, 1) =$ .

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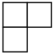
$\mu = (2,1) =$ .

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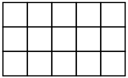

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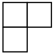
$\mu = (2, 1) =$ .

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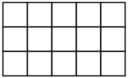

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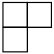
$\mu = (2,1) =$ .

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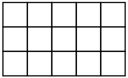

How many rectangles have size n? One for each divisor... Let $\sigma_0(n)$ be the number of divisors of n, then

$$\mathrm{Av}_n\big((2,1)\big) = \sigma_0(n)$$

$$1, 2, 2, 3, 2, 4, 2, 4, 3, 4, \ldots$$

Wilf Equivalence

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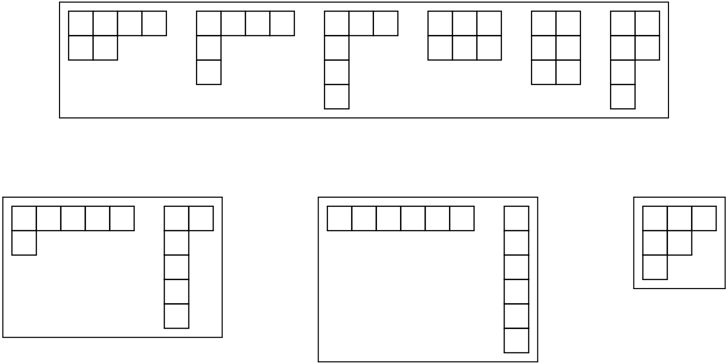
No other pattern is Wilf equivalent to $(2,1)$.

Wilf Equivalence

Wilf classes for $n = 6$:



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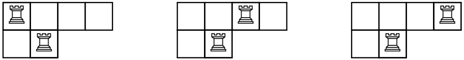

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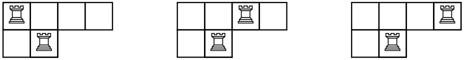

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For any partition $\mu \in \mathbb{P}$ we define its **rook polynomial** to be

$$R_\mu(q) = \sum_{k \geq 0} (\# \text{ of } k \text{ rook-configurations on } \mu) q^k$$

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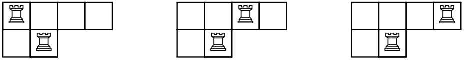

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For example:

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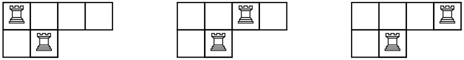

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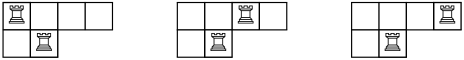

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Two partitions $\mu, \tau \in \mathbb{P}$ are **rook equivalent** if

$$R_\mu(q) = R_\tau(q)$$

i.e., they admit the same number of k-configurations.

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Rook classes for $n = 6$:

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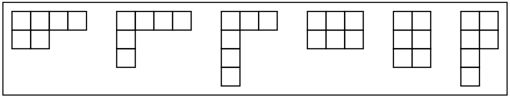

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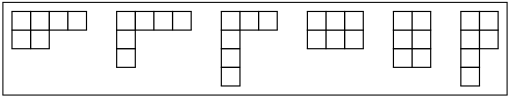

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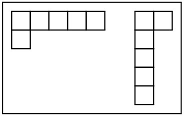

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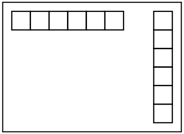

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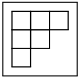

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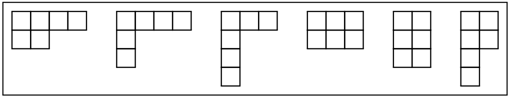

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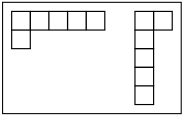

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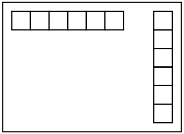

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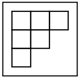

Exactly the same as the Wilf classes!

Wilf and Rook Equivalence

Theorem (Bloom & Saracino (2018))

$$\underbrace{R_\mu(q) = R_\tau(q)}_{\textit{rook equivalence}} \iff \underbrace{|Av_n(\mu)| = |Av_n(\tau)| \textit{ for all } n.}_{\textit{Wilf equivalence}}$$

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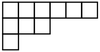

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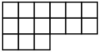

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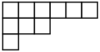

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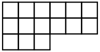

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We can restrict our attention (without loss of generality) to *strict* patterns.

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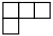
Lets consider the partitions avoiding $\mu = (3, 1)$. 

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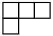
Lets consider the partitions avoiding $\mu = (3, 1)$. 

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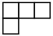
Lets consider the partitions avoiding $\mu = (3, 1)$. 

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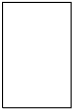

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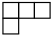
Lets consider the partitions avoiding $\mu = (3, 1)$. 

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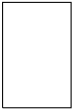

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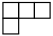
Lets consider the partitions avoiding $\mu = (3,1)$. 

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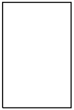

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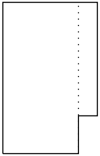

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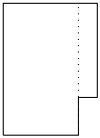

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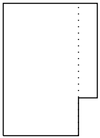

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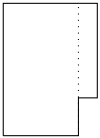

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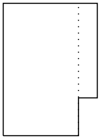

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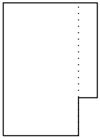

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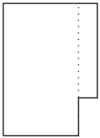

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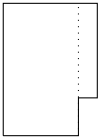

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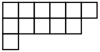

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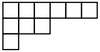

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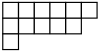

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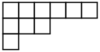

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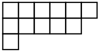

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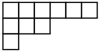

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Let μ be **super-strict**. *Then $A_\mu(z)$ is rational and there exists a recursive algorithm to compute this GF.*

Data

μ	$A_\mu(z)$	OEIS
(2)	$\frac{1}{1-z}$	A000012
(3)	$\frac{1}{(1-z)(1-z^2)}$	A004526
(3,1)	$\frac{1}{(1-z)^2}$	A000027
(4)	$\frac{1}{(1-z)(1-z^2)(1-z^3)}$	A001399
(4,1)	$\frac{z(z^2-z-1)}{(z-1)^3(z+1)^2}$	A117142
(4,2)	$\frac{1-z+z^3}{(1-z)^2(1-z^2)}$	A033638
(5)	$\frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)}$	A001400
(5,1)	$\frac{z(z^5-z^4-z^3+z+1)}{(z-1)^4(z+1)(z^2+z+1)^2}$	A117143
(5,2)	$\frac{-z(z^7-2z^5+z^3+z^2-z-1)}{(z-1)^4(z+1)^2(z^2+z+1)}$	A136185

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Liedahl enumerates metacyclic p-groups and finds that for an odd prime p the number of such groups of order p^n is given by the generating function

$$G(z) = \frac{-z(z^7 - 2z^5 + z^3 + z^2 - z - 1)}{(z-1)^4(z+1)^2(z^2+z+1)}.$$

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Asymptotics

We have

$$
\begin{aligned}
|\mathrm{Av}_n\big((1)\big)| &= 0, & |\mathrm{Av}_n\big((2)\big)| &= 1,\\
|\mathrm{Av}_n\big((2,1)\big)| &= \sigma_0(n), & |\mathrm{Av}_n\big((3)\big)| &= \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} + O(1),\\
|\mathrm{Av}_n\big((3,1)\big)| &= n, & |\mathrm{Av}_n\big((3,2)\big)| &= n\log n + (2\gamma - 2)n + O\left(n^{\frac{131}{416}}\right).
\end{aligned}
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A few new results – asymptotics

Theorem (Bloom & McNew (2019))

Fix $k \geq 1$ *and let* $\mu = (k+1, k, \ldots, 1)$ *be a staircase.*

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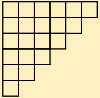
$\mu =$ 

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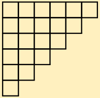
$\mu =$ 

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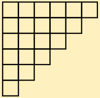
Fix $k \geq 1$ and let $\mu = (k+1, k, \ldots, 1)$ be a staircase. $\mu =$ 

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where $\sigma_k(n) = \sum_{d|n} d^k$.

Proof Idea: Use results of Andrews, Estermann, and Johnson for representations of n as the sum of k products

$$n = \sum_{i=1}^{k} x_i y_i.$$

A few new results – asymptotics

Theorem (Bloom & McNew (2019))

Suppose μ is a strict partition of the form

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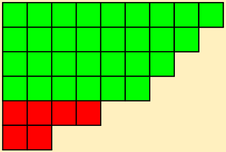

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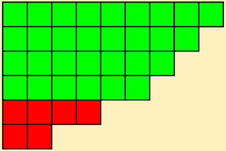

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Then

$$|Av_n(\mu)| \sim \frac{n^{k-1}\log^{\ell} n}{\ell!(k-1)!\prod_{j=0}^{k-\ell-1}(k-\ell-a_j-j)}.$$

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is well known, going back to at least Sylvester (1882).

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If μ is strict with $\mu_1 - \mu_2 = 1$ and $\mu_2 > 0$ then $A_\mu(z)$ is not algebraic.

Conjecture

If μ is strict but not super-strict then $A_\mu(z)$ is not algebraic.

Super-strictness and rationality

Definition

The **southeast border** of μ is the lattice path consisting of the "east" and "north+east" steps tracing along the bottom/right of μ.

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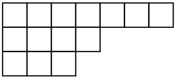
$\mu =$  $\mapsto (e, e, ne, ne, e, e)$

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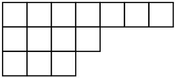
$\mu =$  $\mapsto (e, e, ne, ne, e, e)$

★ A super-strict partition has no consecutive "north+east" steps.

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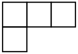
- The partitions avoiding  look like:

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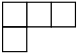
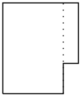
- The partitions avoiding  look like:



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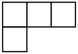
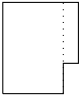
- The partitions avoiding  look like:



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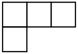
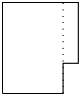
- The partitions avoiding  look like:



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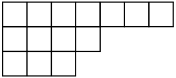
- The partitions avoiding  look like:

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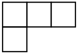
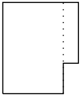
- The partitions avoiding  look like:

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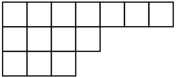
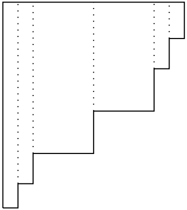
- The partitions avoiding  look like:

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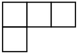
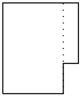
- The partitions avoiding  look like:



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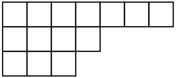
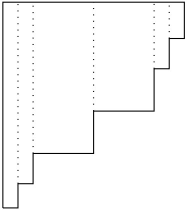
- The partitions avoiding  look like:



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Thank You!