Results in multiplicative combinatorial number theory:

Popular values of the largest prime divisor function

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Distribution of the largest prime divisor

Denote by $P(n)$ the largest prime divisor of n.

What is the distribution of $P(n)$ for $n \in [2, x]$?

Mean value

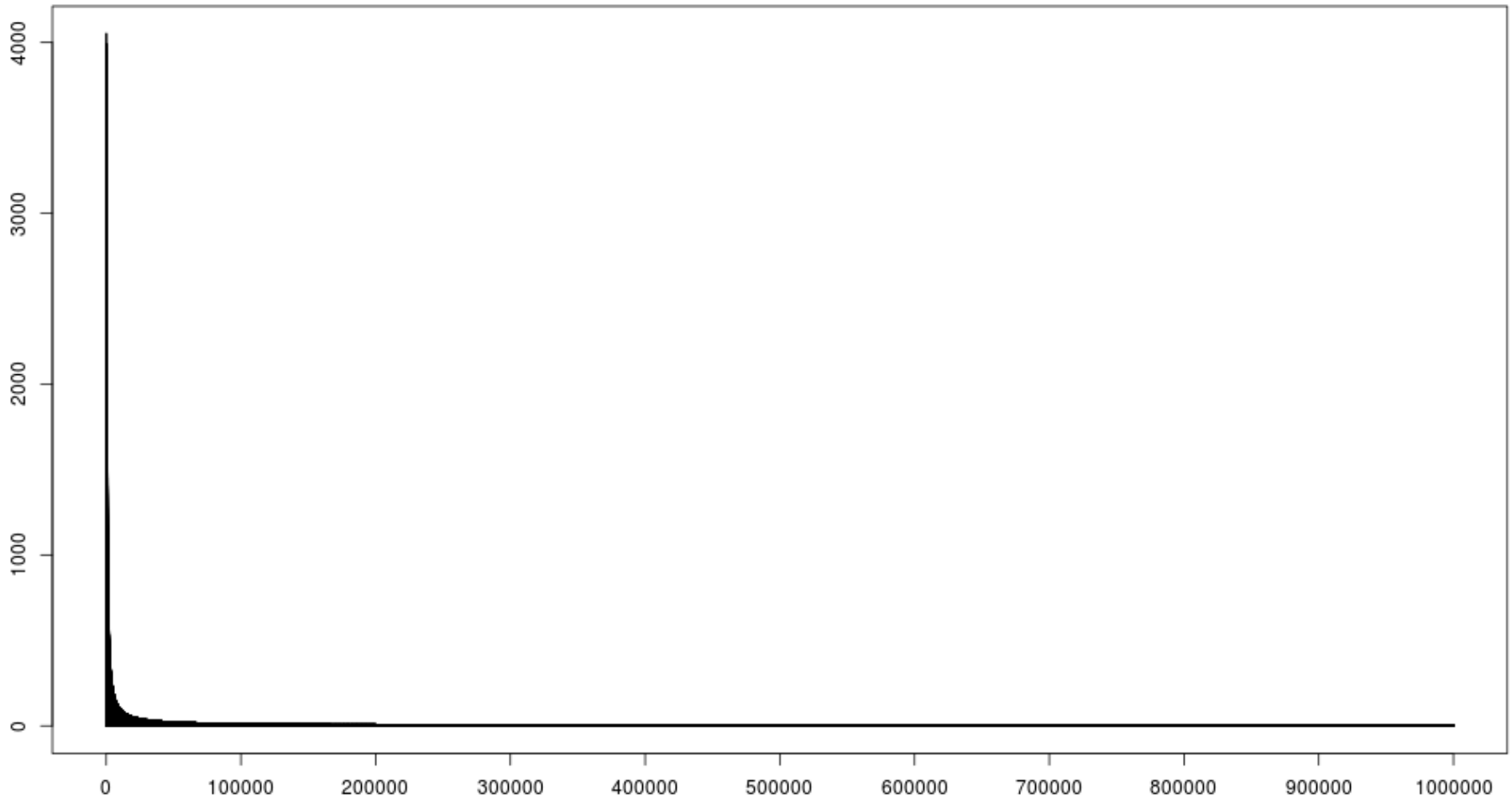
$$(1+o(1))\frac{\pi^2 x}{12\log x} \qquad \text{(Erdős and Alladi, 1977, Kemeny, 1993)}$$

$$\frac{\pi^2 x}{12\log x}+\frac{d_2 x}{\log^2 x}+\ldots+\frac{d_m x}{\log^m x}+O\left(\frac{x}{\log^{m+1} x}\right) \qquad \text{(De Koninck and Ivić, 1984)}$$

Uniformly for all m.

$$d_m = \frac{1}{2^{m+1}}\sum_{j=0}^{m}\frac{(-2)^j\zeta^{(j)}(2)}{j!}. \qquad \text{(Naslund, 2013)}$$

Histogram of $P(n)$ for $n \leq 1{,}000{,}000$



Mean: 64937.45

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$$\frac{\pi^2 x}{12 \log x} + \frac{d_2 x}{\log^2 x} + \ldots + \frac{d_m x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right)$$

Median value

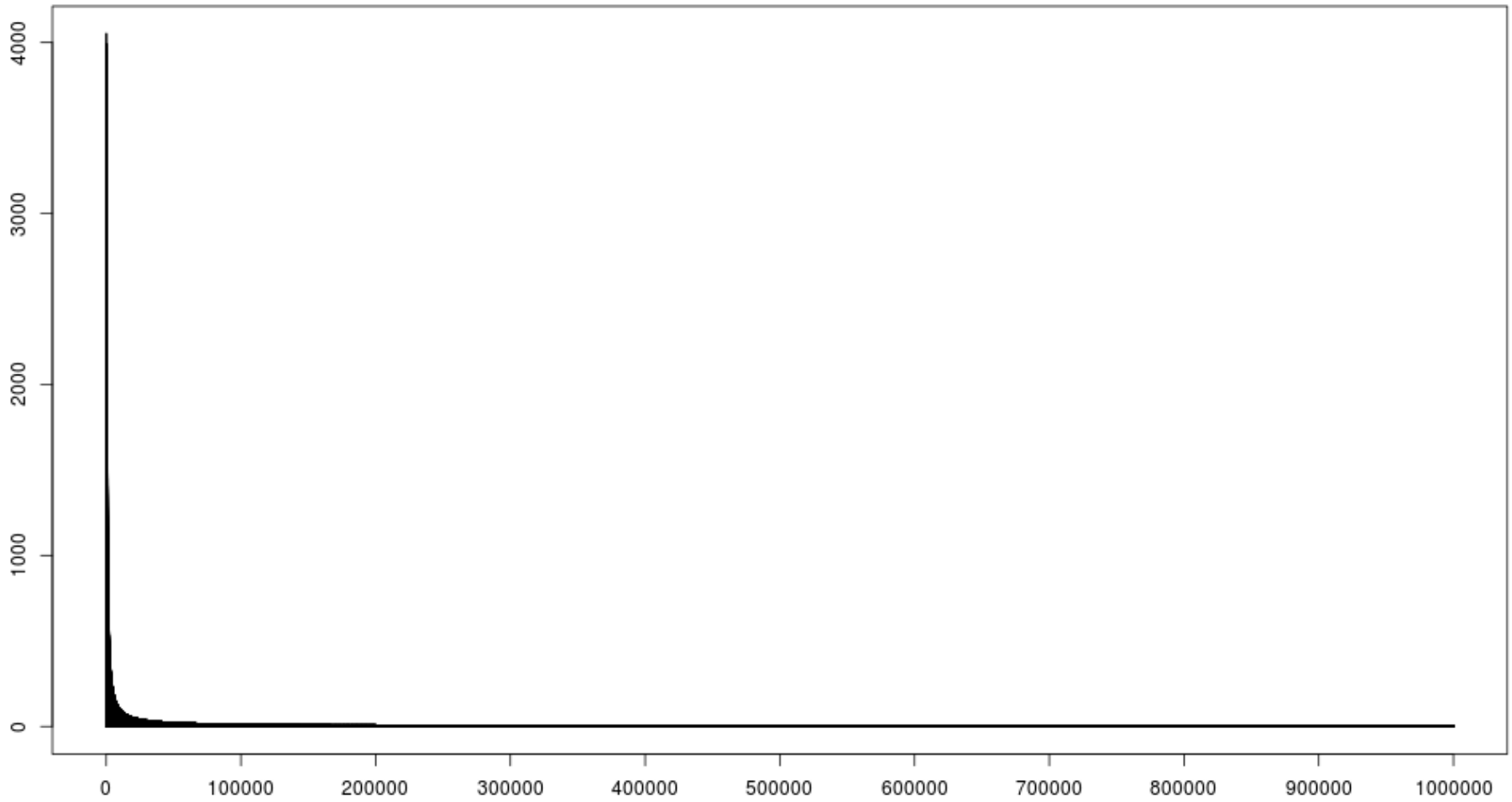
$$x^{\frac{1}{\sqrt{e}}+o(1)}$$

(Selfridge and Wunderlich, 1974)

$$e^{\frac{\gamma-1}{\sqrt{e}}} x^{\frac{1}{\sqrt{e}}} \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

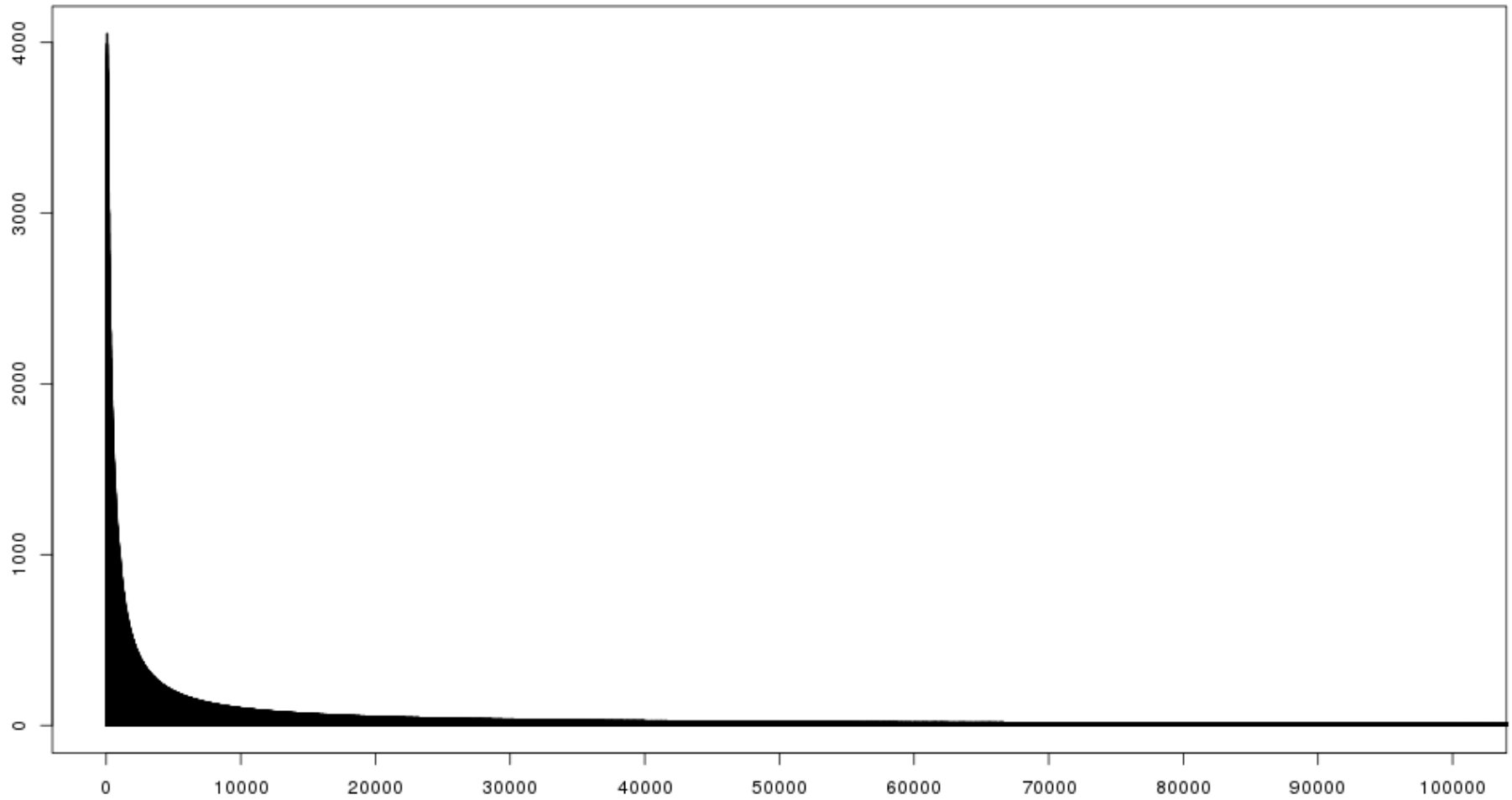
(Naslund, 2013)

Histogram of $P_1(n)$ for $n \leq 1{,}000{,}000$



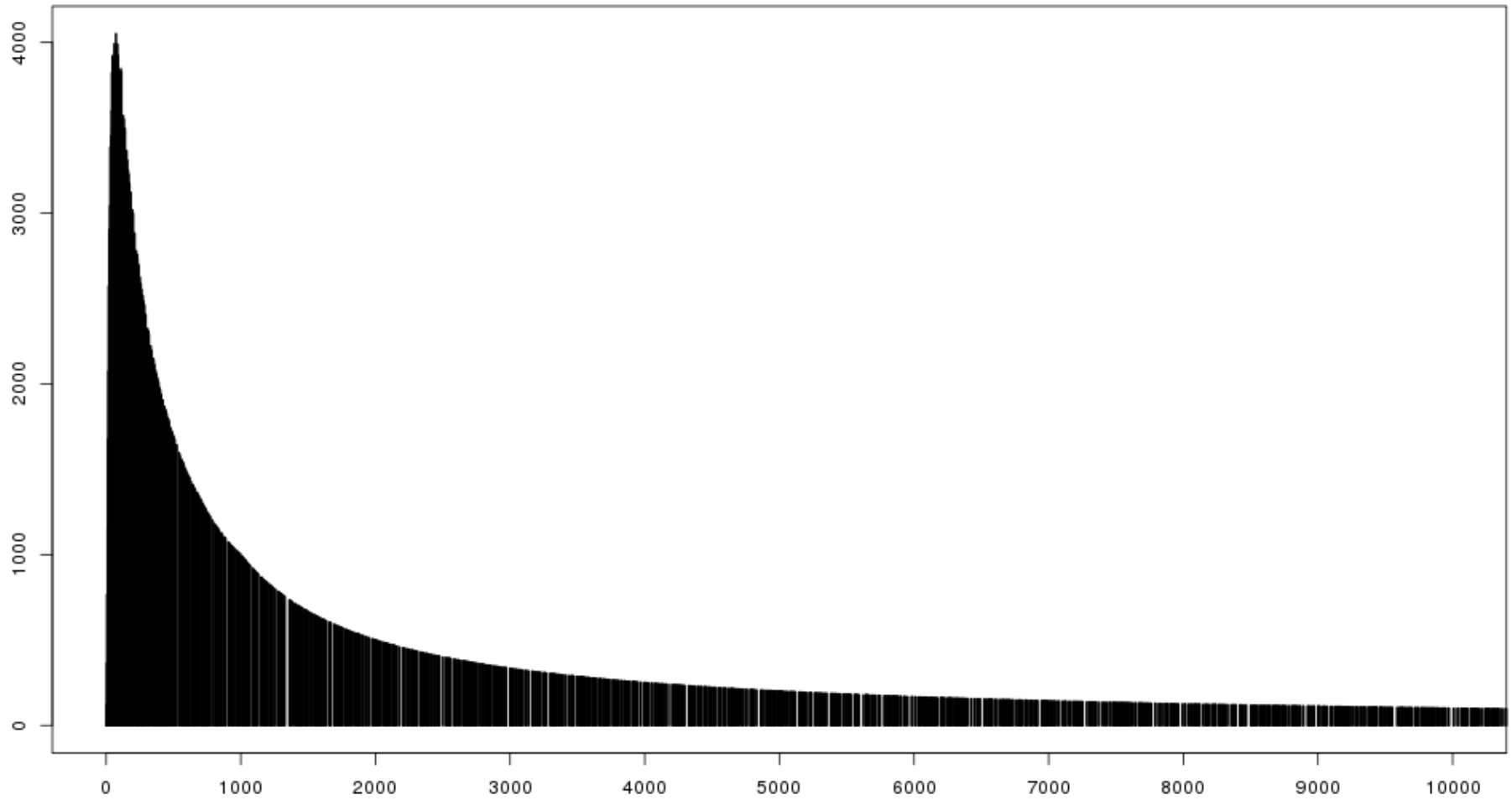
Mean: 64937.45 Median: 3259

Histogram of $P_1(n)$ for $n \leq 1{,}000{,}000$ **Zoomed:10x**



Mean: 64937.45 Median: 3259

Histogram of $P(n)$ for $n \leq 1{,}000{,}000$ **Zoomed:100x**



Mean: 64937.45 Median: 3259 Mode: 73

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Denote by $P(n)$ the largest prime divisor of n.

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$$\frac{\pi^2 x}{12 \log x} + \frac{d_2 x}{\log^2 x} + \ldots + \frac{d_m x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right)$$

Median value

$$e^{\frac{\gamma-1}{\sqrt{e}}} x^{\frac{1}{\sqrt{e}}} \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

Mode

$$e^{\sqrt{\frac{1}{2}\log x(\log\log x + \log\log\log x)} + O\left(\sqrt{\frac{\log x}{\log\log x}}\right)}$$

(de Koninck, 1994)

Counting integers by their largest prime factor

To study the distribution of $P(n)$ we need to count integers up to x whose largest prime divisor is p.

Suppose $n \leq x$ and $P(n) = p$, then $\frac{n}{p} \leq \frac{x}{p}$ and $\frac{n}{p}$ is p-smooth.

An integer is y-**smooth** (or y-**friable**) if all its prime factors are at most y.

Every p-smooth number up to $\frac{x}{p}$ can be multiplied by p to produce a unique integer whose largest prime divisor is p.

$$\#\left\{n \leq x : P(n) = p\right\} = \#\left\{n \leq \tfrac{x}{p} : P(n) \leq p\right\}$$

Psixyology

Denote by $\Psi(x, y)$ the number of y-smooth numbers up to x.

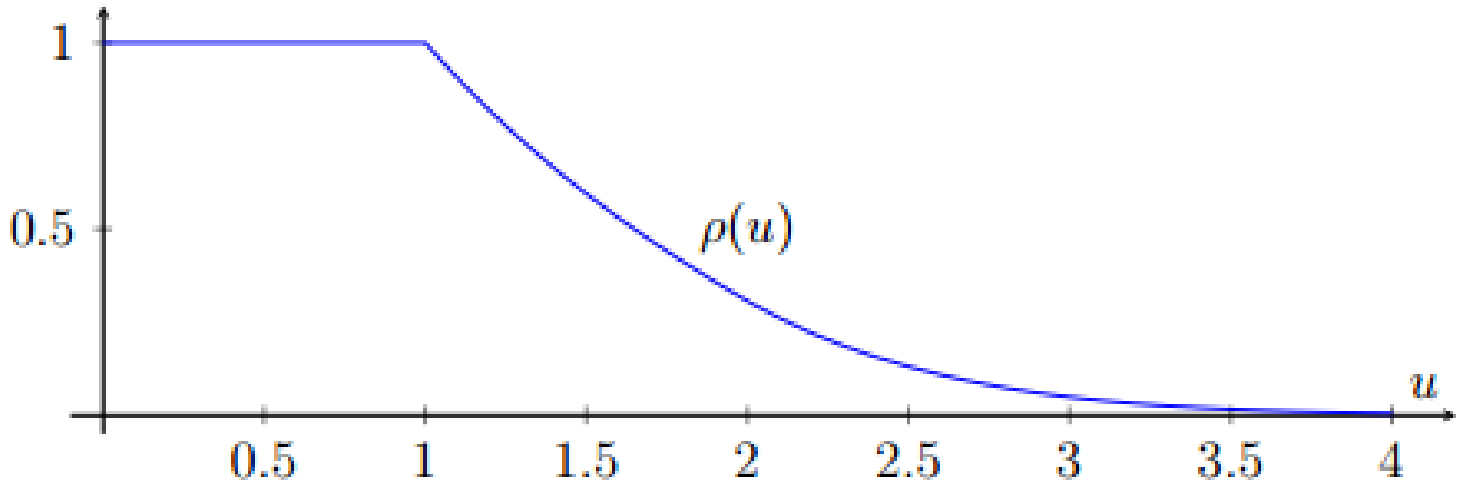
Actuary Karl Dickman (1930) showed that for any fixed u

$$\lim_{x \to \infty} \frac{1}{x} \Psi(x, x^{1/u}) = \rho(u).$$

$\rho(u)$ satisfies a differential delay equation:

$$\begin{aligned} u\rho'(u) + \rho(u-1) &= 0 \\ \rho(u) &= 1 \qquad\qquad 0 \leq u \leq 1 \\ \rho(u) &= 1 - \log u \qquad 1 \leq u \leq 2 \\ &\vdots \end{aligned}$$

Dickman's Rho Function



$$\rho(u) \approx u^{-u}$$

Psixyology

When $1 \leq u \leq 2$, $\rho(u) = 1 - \log u$ and Dickman's result follows from Mertens' theorem

$$\sum_{p<x} \frac{1}{p} = \log\log x + B + o(1)$$

Proof: If $u < 2$ and $n < x$ has a prime divisor greater than $x^{1/u}$ it is unique.

For each $p \in (x^{1/u}, x)$, the number of $n \leq x$ with $p|n$ is $\lfloor \frac{x}{p} \rfloor$.

$$\begin{aligned}
\sum_{x^{1/u}<p\leq x} \left\lfloor \frac{x}{p} \right\rfloor &= x \sum_{x^{1/u}<p\leq x} \frac{1}{p} + O\left(\frac{x}{\log x}\right) \\
&= x\left(\log\log x + B - \log\log x^{1/u} - B + o(1)\right) \\
&= x\log u + o(x)
\end{aligned}$$

So the density of integers without such a factor is $1 - \log u$.

The median largest prime factor

The median largest prime factor m of integers in $[1, x]$ satisfies:

$$\frac{1}{x}\Psi(x, m) = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{1}{x}\Psi(x, x^{1/u}) = \rho(u) \qquad \text{(Dickman)}$$

$$\rho(u) = 1 - \log u \qquad 1 \le u \le 2$$

$$1 - \log u = 1/2 \qquad \Rightarrow u = \sqrt{e}$$

$$m = x^{\frac{1}{\sqrt{e}} + o(1)}$$

Psixyology

De Bruijn (1951) shows that if $u = \frac{\log x}{\log y}$

$$\Psi(x, y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right)$$

as long as $1 \leq u \leq (\log y)^{3/5-\epsilon}$ or equivalently $y > \exp((\log x)^{5/8+\epsilon})$.
He also gives an estimate for $\rho(u)$:

$$\rho(u) = (1 + o(1))\frac{1}{\sqrt{2\pi u}}\exp\left\{\gamma + u\xi(u) + \int_0^{\xi(u)} \frac{e^s - 1}{s} ds\right\}$$

where $\xi(u)$ is the positive solution to

$$e^{\xi(u)} = 1 + u\xi(u).$$

Psixyology

Hildebrand (1986) shows that De Bruijn's approximation

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right)$$

holds for $\exp((\log\log x)^{5/3+\epsilon}) < y \leq x$.

Alladi (1984) shows that

$$\rho(u) = \left(1 + O\left(\frac{1}{u}\right)\right)\sqrt{\frac{\xi'(u)}{2\pi}}\exp\left\{\gamma + u\xi(u) + \int_0^{\xi(u)} \frac{e^s - 1}{s}ds\right\}.$$

Mode of $\{P(n)|n \leq X\}$

Use Hildebrand and Alladi's results to approximate $\Psi\left(\frac{x}{p}, p\right)$.

Let $Q(x)$ be the prime p which maximizes $\Psi\left(\frac{x}{p}, p\right)$.
$Q(x)$ is the most popular large prime divisor on the interval $[2, x]$.

Theorem

The mode, $Q(x)$, satisfies

$$Q(x) = e^{\sqrt{\upsilon(x)\log x} + O((\log\log x)^{1/4})}$$

where $\upsilon(x)$ is the unique solution to

$$e^{\upsilon(x)} = 1 + \sqrt{\upsilon(x)\log x - \upsilon(x)^2}.$$

$$\upsilon(x) = \tfrac{1}{2}\log\log x + \tfrac{1}{2}\log\log\log x - \tfrac{1}{2}\log 2 + o(1)$$

Popular Primes

Consider the largest prime divisor of the first few integers:

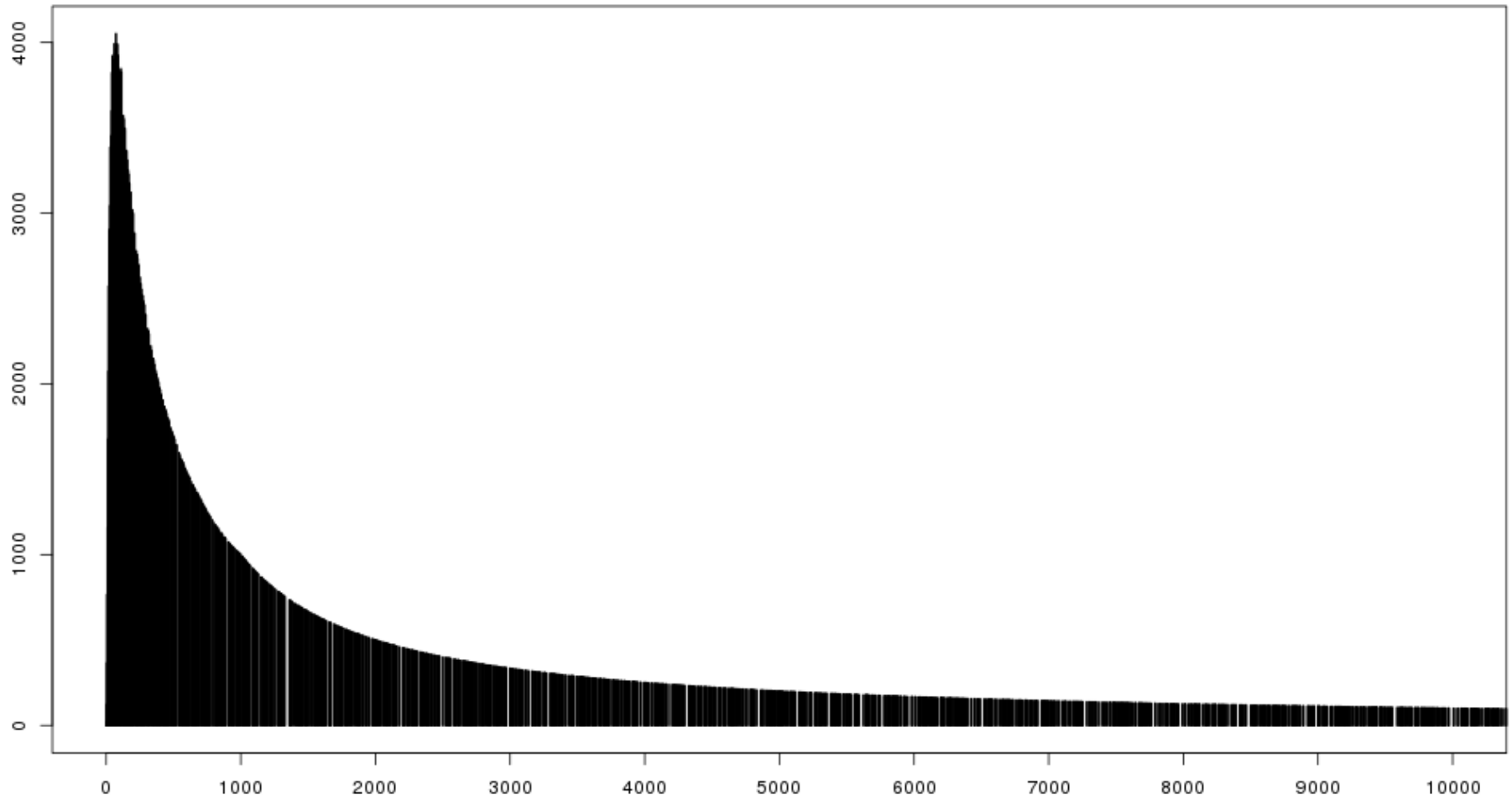
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	3	2	5	3	7	2	3	5	11	3	13	7	5	2	17	3

Prime	First popular	Last popular
2	2	17
3	3	119
5	45	279
7	70	1858
13	1456	5471

Call a prime p a **Popular Prime** if there exists an N such that

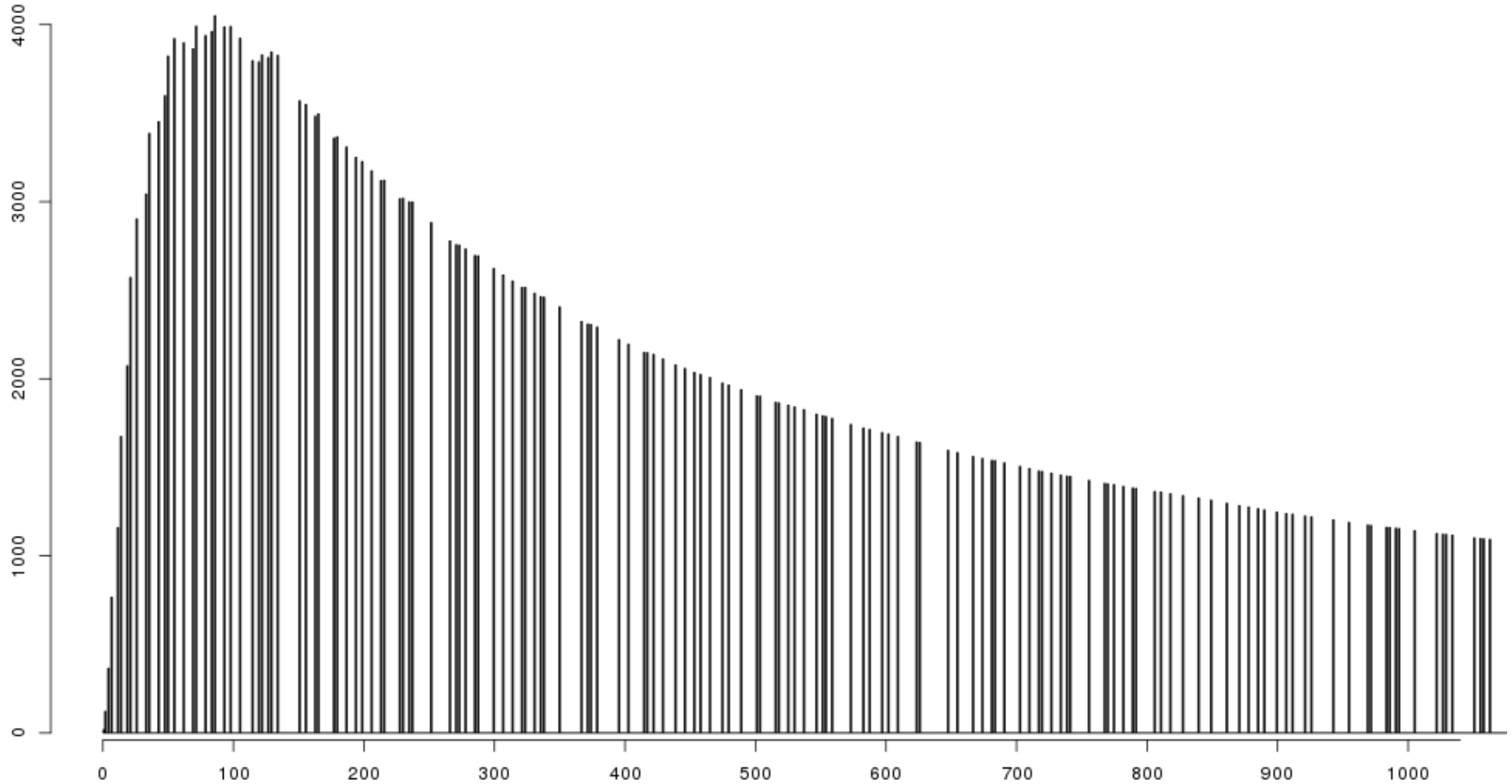
$$p = \text{mode}\{P(n) : n \in [2, N]\}.$$

Histogram of $P(n)$ for $n \leq 1{,}000{,}000$ **Zoomed:100x**



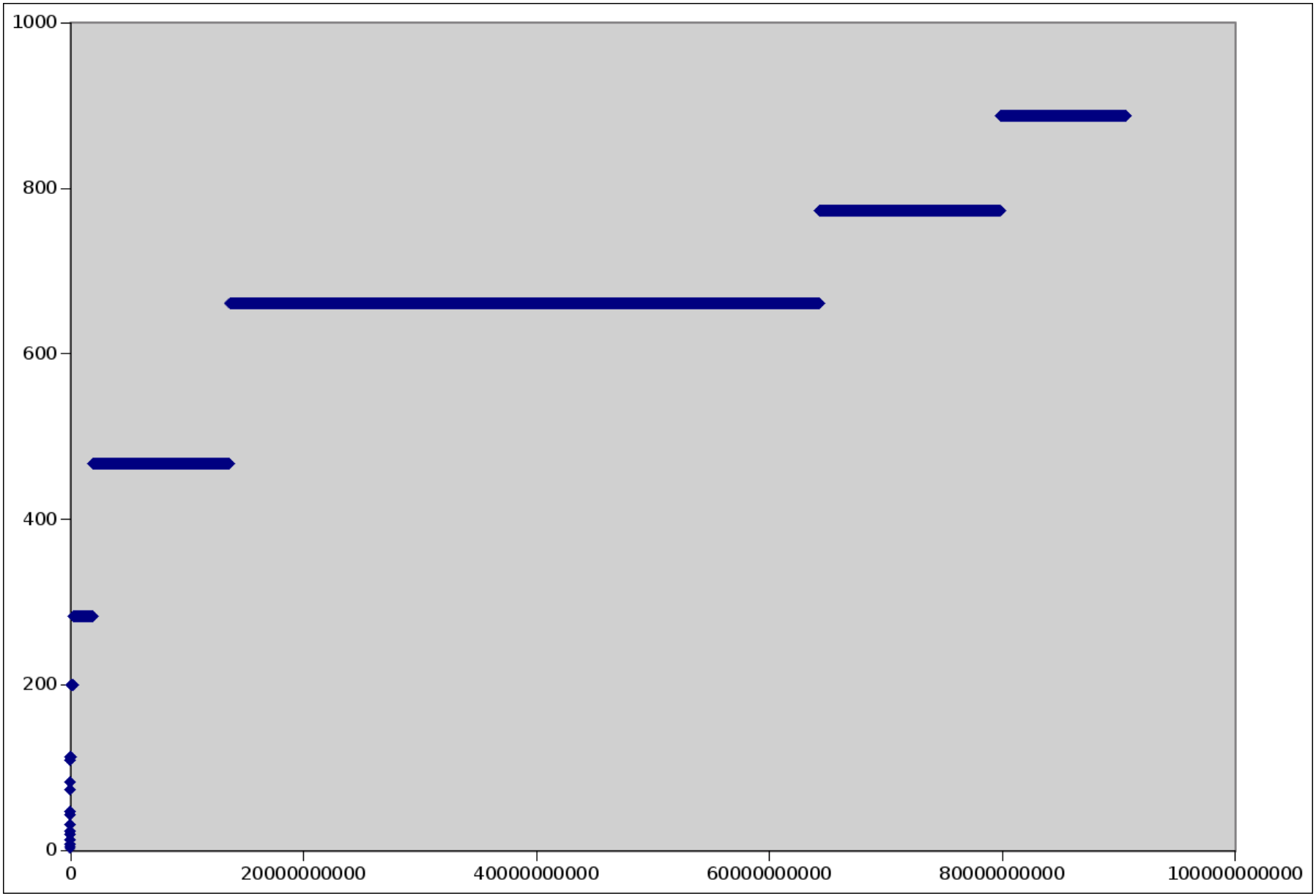
Mean: 64937.45 Median: 3259 Mode: 73

Histogram of $P(n)$ for $n \leq 1{,}000{,}000$ **Zoomed:1000x**

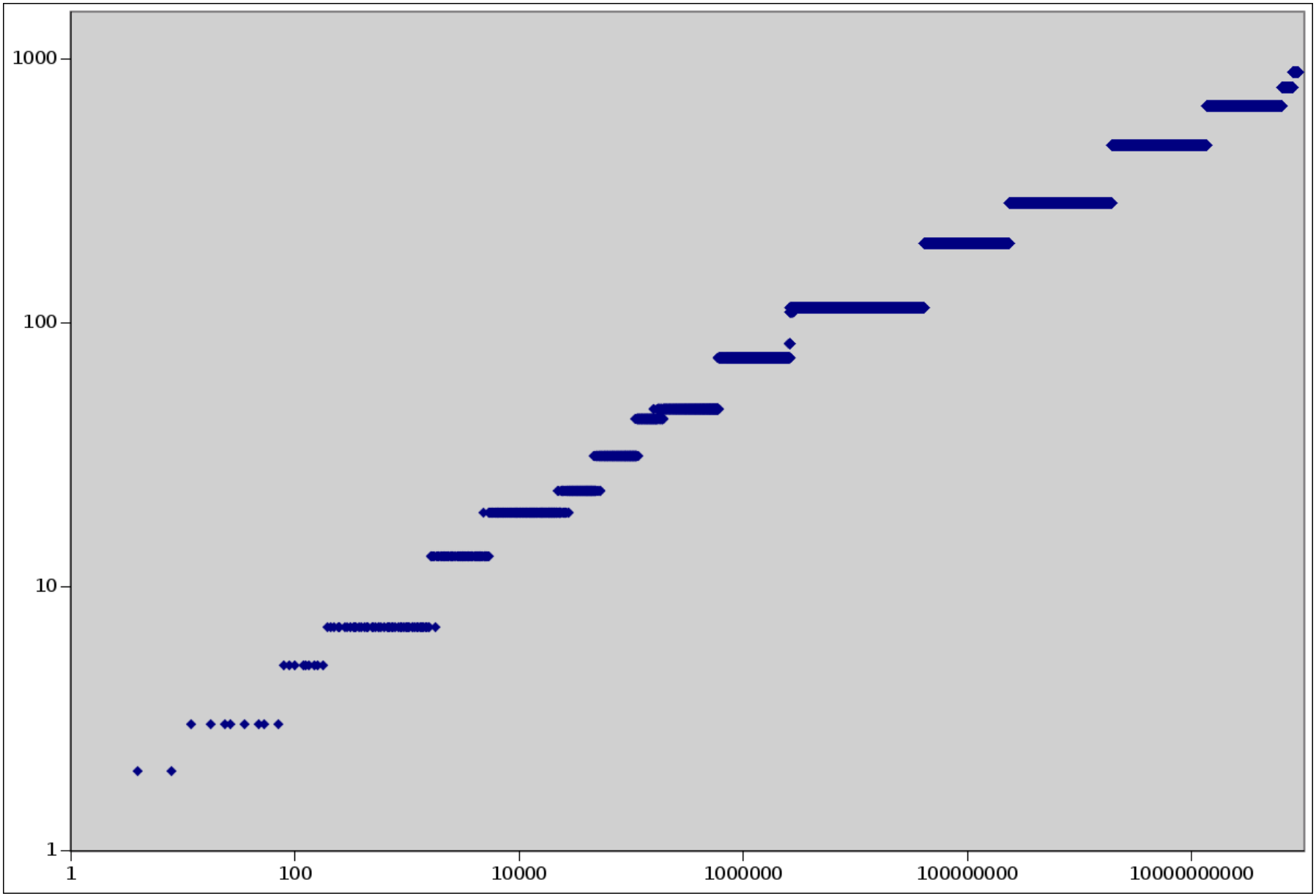


Mean: 64937.45 Median: 3259 Mode: 73

Most Popular Largest Prime Divisor up to 90 Billion



Most Popular Largest Prime Divisor up to 90 Billion



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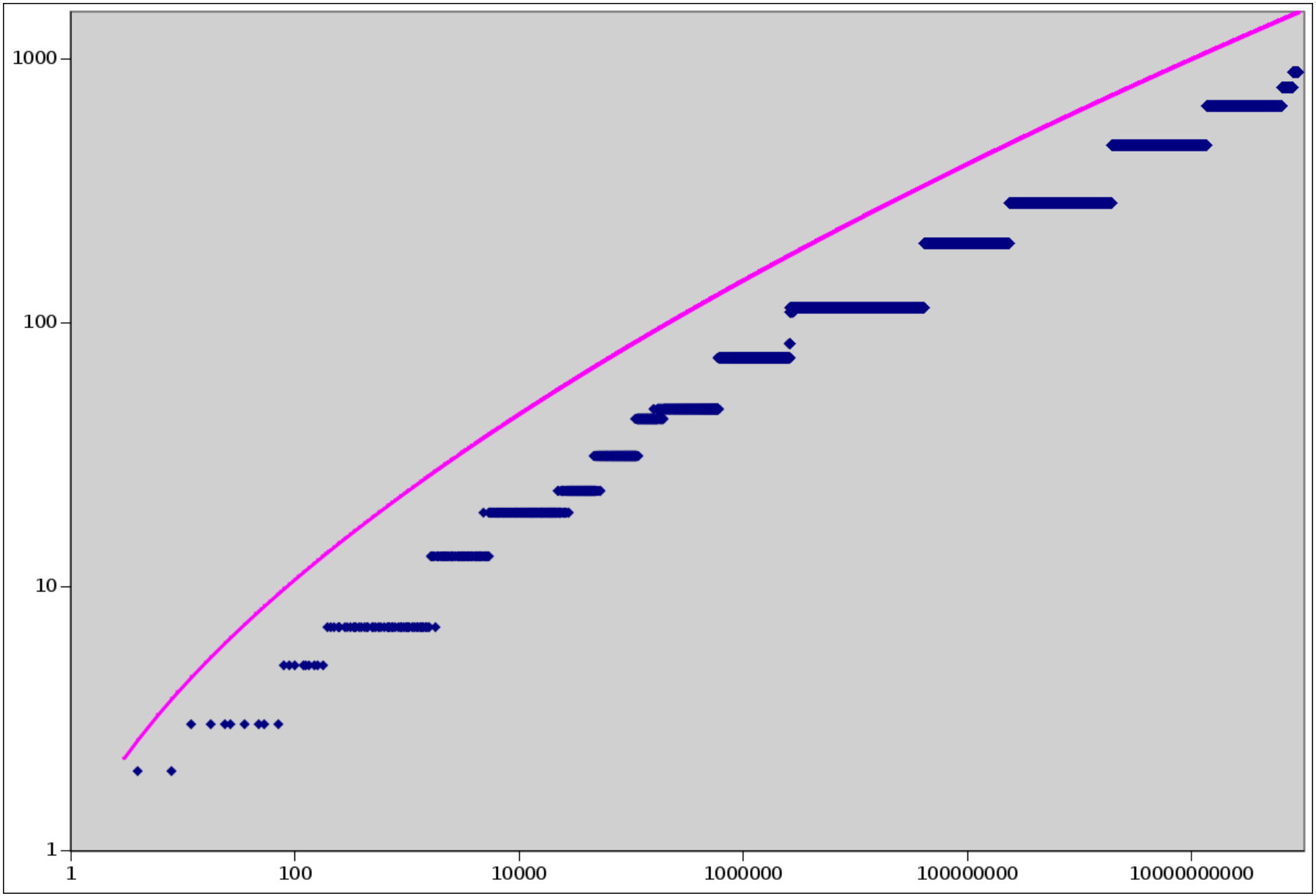


The primes popular for some $x \leq 7 \times 10^{13}$

Popular prime	First Popular	Count
2	2	1
3	12	4
5	80	12
7	196	21
13	1638	72
19	4864	141
23	22425	365
31	46500	565
43	109779	965
47	158625	1224
73	603564	2880
83	2552416	7490
109	2620142	7622
113	2627250	7636
199	41163747	48358
283	237398795	161542
467	1966466950	697876
661	13690729828	2760914
773	64322158656	8354318
887	79838739611	9754754
1109	220355987735	20284681
1129	232268774850	21082413
1327	618745972214	43030538
1627	1882062476406	96835109
2143	9607713772982	318536261
2399	19364051829855	534252391
2477	26393150937218	672026919
2803	37636607806688	873944931

Which primes are popular?

Write: $p_n = n$th prime
$d_n = p_{n+1} - p_n$ (nth prime gap)

Theorem

For sufficiently large n, if $d_{n-1} > d_n$, then p_n is not popular.

Theorem

If p_n and p_{n+k} are both popular primes then they satisfy

$$\frac{p_{n+k} - p_n}{k} = \log p_{n+k} + O(\log \log p_{n+k}).$$

How many popular primes are there?

Very average spacing + Brun's Sieve:

Theorem

The number of popular primes up to x is at most $\frac{x}{(\log x)^{4/3+o(1)}}$.

By the prime number theorem there are about $\frac{x}{\log x}$ primes up to x.
However unlike the primes...

Corollary

The sum of the reciprocals of the popular primes converges.

Recall $Q(x) = e^{\sqrt{v(x)\log x}+O((\log\log x)^{1/4})}$.

Theorem

The number of popular primes up to x is asymptotically at least

$$\frac{\log x}{(\log\log x)^{1/4}}.$$

An application: factoring

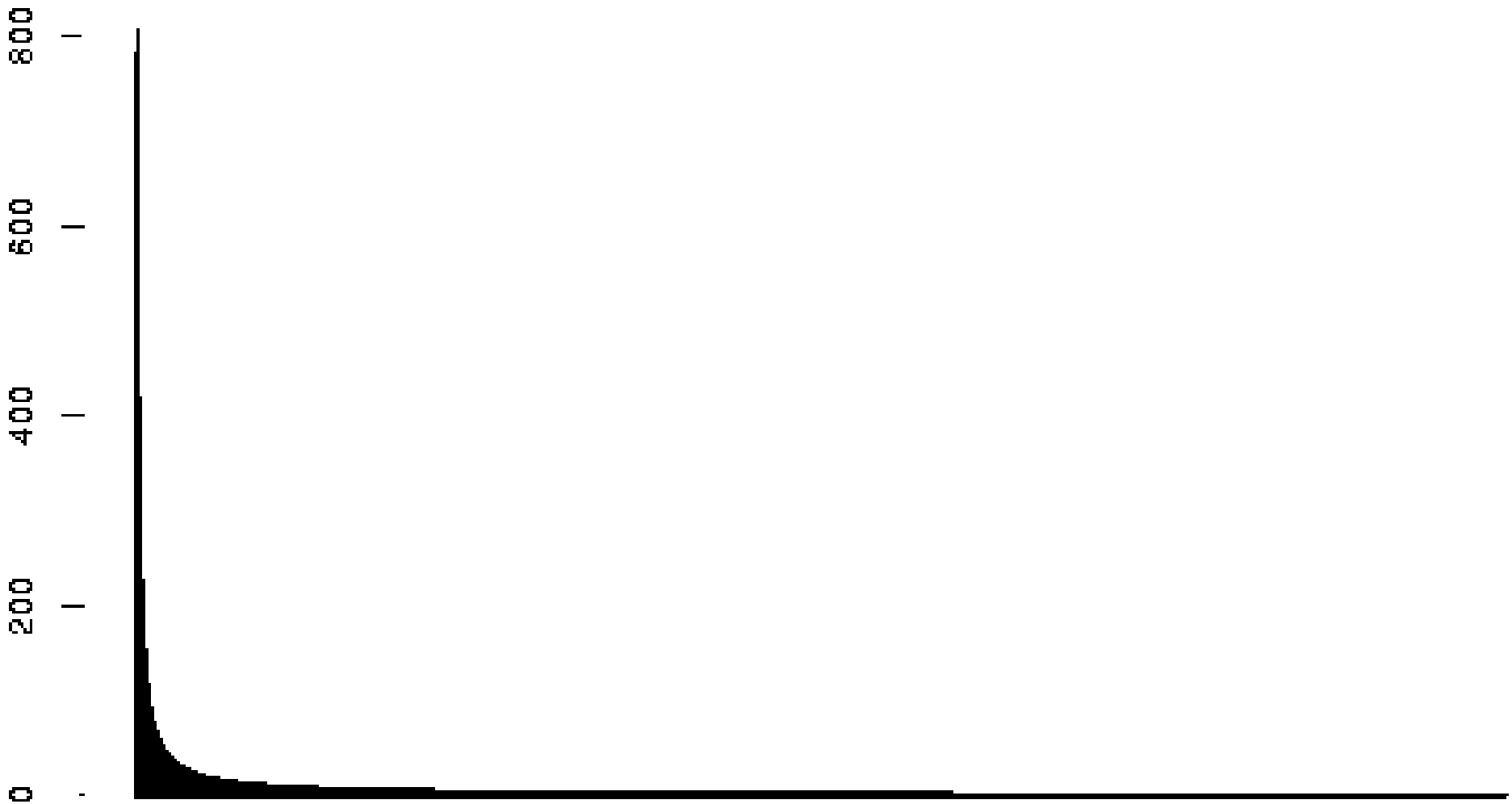
A critical step in many factoring algorithms (Dixon's random squares, quadratic sieve, number field sieve...) is to generate integers $a_1, a_2 \ldots$ such that $b_i \equiv a_i^2 \pmod{n}$ is y-smooth until some product of the a_i is a square mod n. (Square dependence)

How to pick the smoothness bound y?

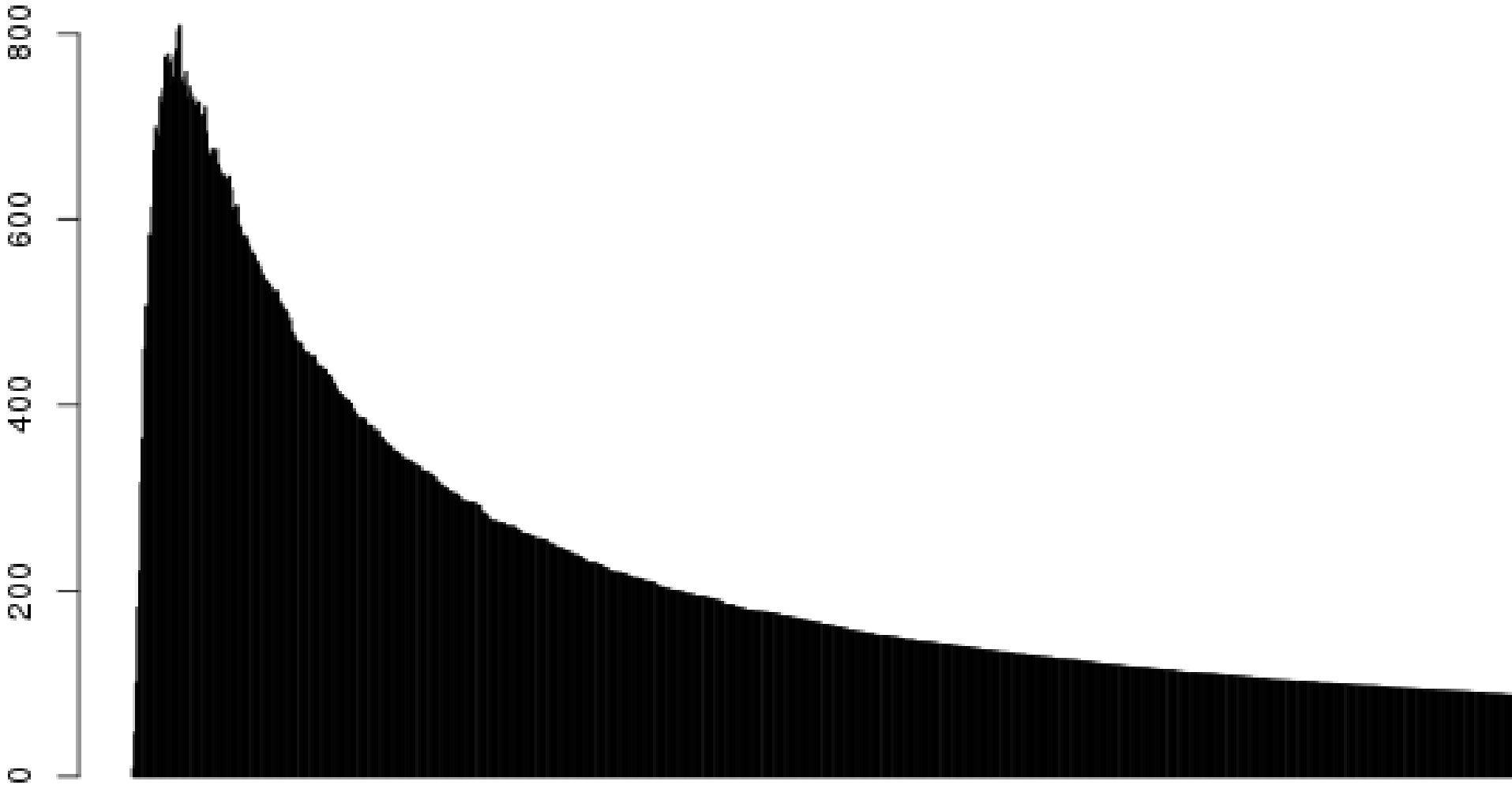
- Expected number of trials to pick a y-smooth integer is $\frac{x}{\Psi(x,y)}$.
- Having $\pi(y) + 1$ integers forces a square dependence.

Optimal value of y minimizes the expression $\frac{x\pi(y)}{\Psi(x,y)} \approx \frac{xy}{\Psi(x,y)}$.
(Maximizes $\frac{\Psi(x,y)}{y}$.)

Distribution of $\Psi(x, y)/y$ for $x = 1{,}000{,}000$



$\Psi(x, y)/y$ for $x = 1{,}000{,}000$ **Zoomed:100x**



Peak: 113

Optimizing factoring algorithms

Croot, Granville, Pemantle and Tetali showed (2008, Annals of Math) that the optimal smoothness bound y_0, which maximizes $\Psi(x,y)/y$, satisfies

$$y_0 = e^{\sqrt{\frac{1}{2}\log x(\log\log x+\log\log\log x-\log 2+o(1))}}.$$

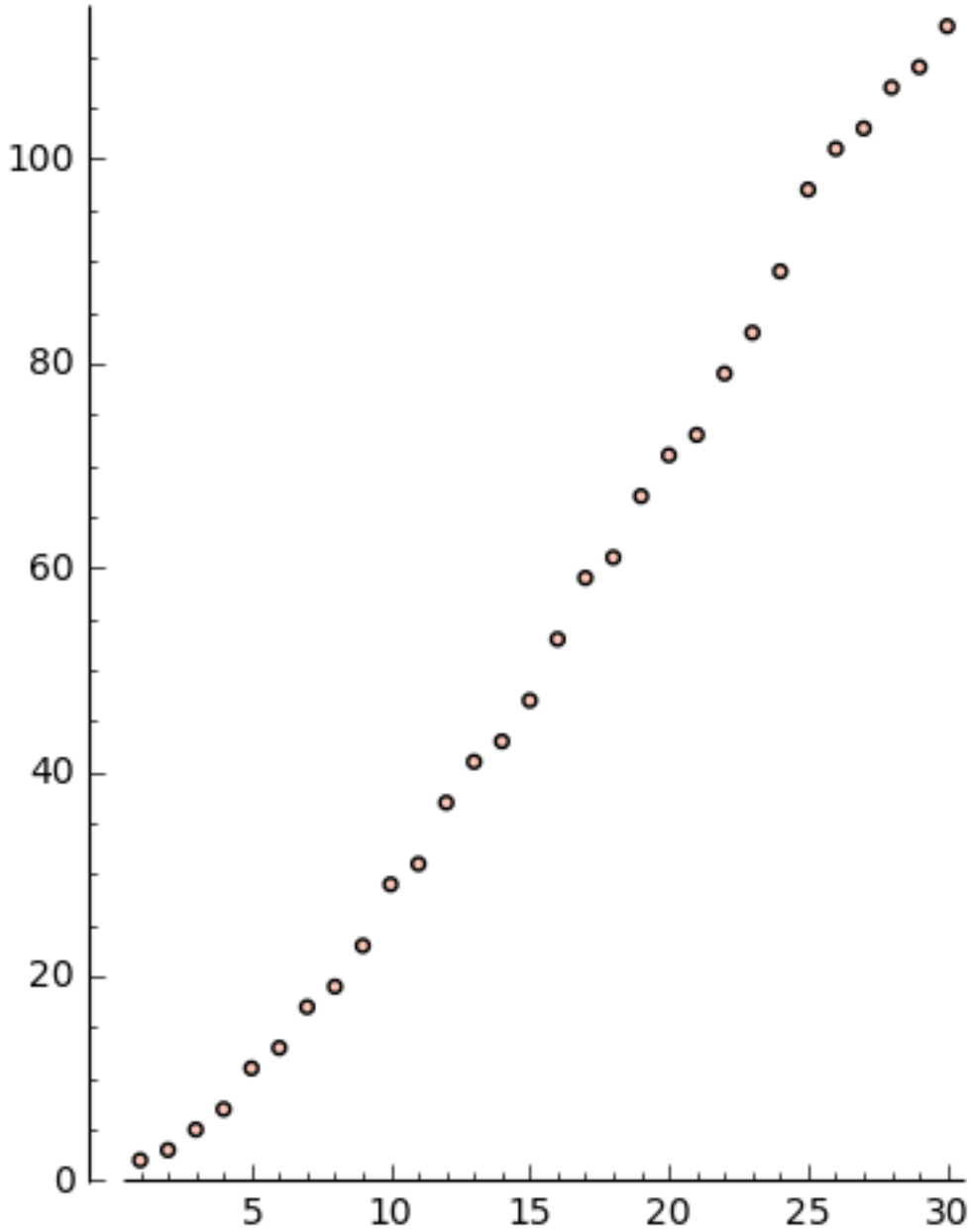
Future work: Compare the set of "Popular Primes" to the set of those "**Fast Primes**" p which minimize $\Psi(x,p)/p$ for some value of x.

The convex primes

Consider the collection of points (n, p_n) in $\mathbb{R}^2$. (The prime number graph) Generally (but somewhat erratically) curve upward, $p_n \sim n \log n$.

Take the convex hull of these points. Say that a prime p_n is a **convex prime** if (n, p_n) is a vertex point of this convex hull.

The prime number graph



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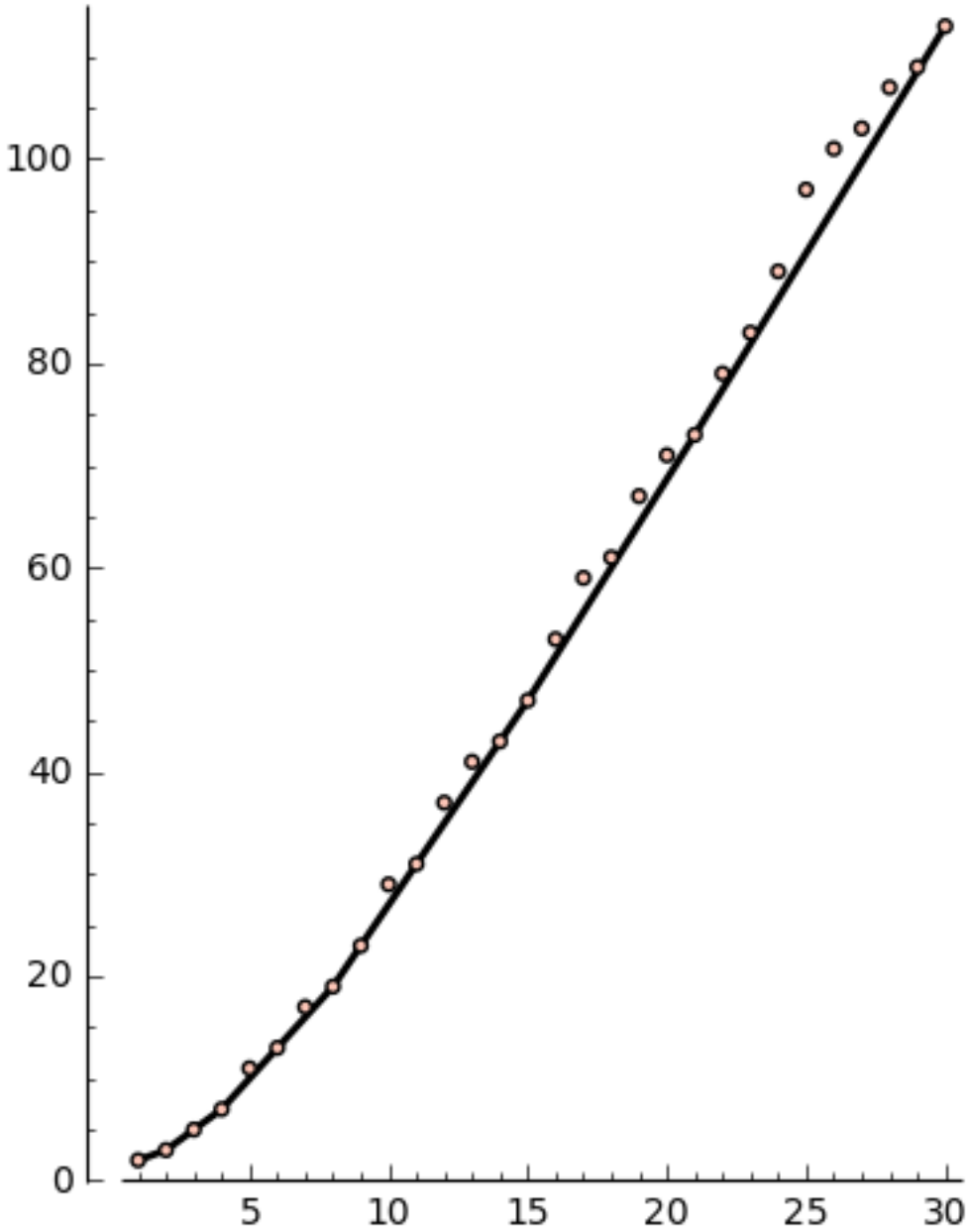

Midpoint convex primes

p_n is a convex prime if and only if for every $j < n$ and $k > n$ the line segment connecting (j, p_j) to (k, p_k) passes above the point (n, p_n).

A weaker requirement would ask that for each $i < n$ the line segment from $(n-i, p_{n-i})$ to $(n+i, p_{n+i})$ pass through or above the point (p_n, n).
Call such primes **midpoint convex**.

$$\forall i < n \quad p_{n-i} + p_{n+i} \geq 2p_n$$

Every convex prime is midpoint convex. Pomerance uses this and the geometry of the prime number graph to prove there are infinitely many of each.

Discussed in problem **A14** in Guy's Unsolved Problems in Number Theory

Popular Primes and Convex Primes

The first few popular primes are:

$$2, 3, 5, 7, 13, 19, 23, 31, 43, 47, 73, 83, 109, 113, 199, 283, 467,$$
$$661, 773, 887, 1109, 1129, 1327, 1627, 2143, 2399, 2477, 2803$$

The first few convex primes are:

$$2, 3, 7, 19, 47, 73, 113, 199, 283, 467, 661, 887, 1129, 1327, 1627, 2803$$

Thus far every convex prime is a popular prime.
(Also $5, 13, 23, 31, 43$, lie on the convex hull but are not vertex points.)

Every known popular prime is midpoint convex besides 773.
Popular primes must satisfy $p_{n-1} + p_{n+1} > 2p_n$.

Counting convex primes

The count of the convex primes up to x is at least $\exp(c(\log x)^{3/5})$ for some constant c. (Pomerance)

Assuming the Riemann hypothesis this can be improved to $x^{1/4}/\log^{3/2} x$.

A result of Erdős and Prachar implies that the count is $o(\pi(x))$.

Theorem

The count of the convex primes up to x is $O\left(\frac{x^{2/3}}{\log^{2/3} x}\right)$.

Counting convex primes

Theorem

The count of the convex primes up to x is $O\left(\frac{x^{2/3}}{\log^{2/3} x}\right)$.

Proof:

Using the prime number theorem we can show that the slope of the convex hull following a vertex point (n, p_n) is $\log n + \log\log n - 1 + o(1))$.

Let $p_{i_1}, p_{i_2}, \ldots$ be the consecutive convex primes in the interval $(\frac{1}{2}x, x]$.

The slopes between consecutive convex primes are increasing rational numbers

$$s_j = \frac{p_{i_{j+1}} - p_{i_j}}{i_{j+1} - i_j}.$$

Because $p_{i_j} \in (\frac{1}{2}x, x]$, we have $\frac{1}{2}\pi(x) < \pi\left(\frac{1}{2}x\right) < i_j \leq \pi(x)$ so each p_{i_j} is contained in some interval of length $\log 2 + o(1)$.

Counting convex primes

Theorem

The count of the convex primes up to x is $O\left(\frac{x^{2/3}}{\log^{2/3} x}\right)$.

Proof Continued:

$$s_j = \frac{p_{i_{j+1}} - p_{i_j}}{i_{j+1} - i_j}$$

$s_j < s_{j+1} < \ldots$ and all are contained in an interval of length $\log 2 + o(1)$.

$i_j - i_{j-1} = k \Longrightarrow$ at most $k(\log 2 + o(1))$ possible values of $p_{i_j} - p_{i_{j-1}}$.

For each k, there are $O(k)$ convex primes p_{i_j} with $i_j - i_{j-1} = k$, or $O(K^2)$ convex primes which follow a gap of at most K convex primes.

The number of consecutive convex primes with $i_j - i_{j-1} > K$ is $O\left(\frac{x}{K \log x}\right)$.

Optimizing K we find $K = (x/\log x)^{1/3}$. So the count of convex primes in $(\frac{1}{2}x, x]$ is $O\left(\frac{x^{2/3}}{\log^{2/3} x}\right)$. Sum dyadically to complete the proof.

Generalizations

The most popular kth-largest prime divisor ($k \geq 2$) is always 3. (de Koninck)

The most common largest part of factorizations in more general settings. (Rings of integers in a number field, polynomials over a finite field...)

When factoring polynomials of degree n over $\mathbb{F}_q$, which polynomials (degrees) show up most often as the irreducible factor with highest degree?

All polynomials with the same degree occur with the same frequency: No analogue of a popular polynomial.

Additive factorizations: partitions

A **partition** of a positive integer n is a way of writing n as a sum of positive integers. (Order doesn't matter)

For example: $7=3+2+1+1$

Over all partitions of n, what is the most popular largest component?

Let $K(n)$ denote the most popular largest component of the partitions of n.

Theorem (Auluck, Chowla, Gupta, 1942)

$$n^{1/2} < K(n) < \frac{\sqrt{6}}{\pi} n^{1/2} \log n^{1/2}$$

Additive factorizations: partitions

Let $K(n)$ denote the most popular largest component among the partitions of n.

Theorem (Erdős, 1946)

$$K(n) = \frac{\sqrt{6}}{\pi} n^{1/2} \log\left(\frac{\sqrt{6}}{\pi} n^{1/2}\right) + o(n^{1/2})$$

Theorem (Szekeres, 1953)

$K(n)$ is the closest integer to

$$\frac{\sqrt{6}}{2\pi} n^{\frac{1}{2}} \log\left(\frac{6}{\pi^2} n\right) - \frac{6}{\pi^2}\left(\frac{1}{16}\log^2\left(\frac{6}{\pi^2} n\right) - \frac{3}{4}\log\left(\frac{6}{\pi^2} n\right) - \frac{3}{2}\right) - \frac{1}{2} + O\left(\frac{\log^4 n}{n^{\frac{1}{4}}}\right)$$

The end

Thank you