The density of covering numbers

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Erdős and some conjectures

- Erdős introduced covering systems in 1950, introduced the Minimum Modulus Problem: "Are there distinct covering systems whose smallest modulus exceeds M for any M?"
- Many additional conjectures an problems of Erdős and others.
- Is there a distinct covering system with only odd moduli?
  (Erdős-Selfridge conjecture)

Exciting Developments

- Minimum Modulus Problem
  - Hough (2015): M cannot be arbitrarily large.
  - Owens (2014), and Balister, Bollobás, Morris, Sahasrabudhe, Tiba (2018):

$$42 \le m_1 < 616000.$$

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$$42 \leq m_1 < 616000.$$
- The Erdős-Selfridge conjecture remains open.
  - Hough, Nielsen (2017) a modulus is divisible by either 2 or 3.
  - Balister et al (2018): a modulus is divisible by 2, 9 or 15.

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Haight (1979), answering an Erdős problem, defines **covering numbers**: integers whose divisors form the moduli of a distinct covering system.

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Any multiple of a covering number is also covering, so call n a **primitive covering number** if it is a covering number but no proper divisor of n is.

Primitive covering numbers (A160559): $12, 80, 90, 210, 280, 378, 448, 1386\ldots$

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Erdős-Selfridge Conjecture: There are no odd covering numbers. After Bellobas et al, every covering number is a multiple of 2, 9 or 15.

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Define the "abundancy index" $h(n) := \frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}$.

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An arithmetic progression mod d “covers” a set of integers of density $\frac{1}{d}$, so a necessary (not sufficient) condition for n to be a covering number is

$$\sum_{d|n,\ d>1} \frac{1}{d} \geq 1$$

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Primitive Covering Numbers

Theorem (Sun, 2007)

There are infinitely many primitive covering numbers $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, *where* $\alpha_k = 1$, $\alpha_{k-1} = \left\lfloor \frac{p_k - 1}{p_{k-1} - 1} \right\rfloor$

$$p_i \;=\; 1 + \tau\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}\right) \qquad (1 < i < k)$$

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Harrington, Jones, Phillips (2017) find infinitely many more of form $2^{\alpha} p^{\beta} q$.

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Notation Summary

- $\tau(n)$: number of divisors,
- $P^+(n), P^-(n)$: largest, smallest prime factor
- $\mathcal{A}$, $\mathcal{C}$, sets of abundant and covering numbers
- $\mathcal{P}_\mathcal{A}, \mathcal{P}_\mathcal{C}$ their primitive subsets
- $\mathcal{S}(x) := \#\{n \le x \mid n \in \mathcal{S}\}$ the counting function of the set S.
- $d(\mathcal{S}) := \lim_{x\to\infty} \frac{\mathcal{S}(x)}{x}$ denotes the natural density of $\mathcal{S}$ if it exists.

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On average, $\frac{1}{x}\sum_{n\le x}\tau(n) \approx \log x$, while $\frac{1}{x}\sum_{n\le x}P^+(n) \approx \frac{\pi^2}{12}\frac{x}{\log x}$.

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- **Many divisors:** $\Delta(x,y)$ counts integers $\le x$ with $\tau(n) > y$

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So $\mathcal{P}_{\mathcal{C}}(x) \le \psi(x,y) + \Delta(x,y) \approx \dfrac{x}{\exp\left(\frac{\log x}{\log y}\log\frac{\log x}{\log y}\right)} + \dfrac{x}{\exp\left(\frac{\log y}{\log 2}\log\log y\right)}.$

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- *The reciprocal sum* $\sum_{c\in\mathcal{P}_{\mathcal{C}}} \frac{1}{c}$ *converges.*

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- *The reciprocal sum* $\sum_{c \in \mathcal{P}_{\mathcal{C}}} \frac{1}{c}$ *converges.*
- *The density* $d(\mathcal{C})$ *of covering numbers exists.*

Bounding $d(\mathcal{A})$ - the Behrend–Deléglise Approach

Define $\quad A_y(x) = d(\{n : P^-(n) > y,\ h(n) > x\})$

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Bound the tail sum trivially: $\displaystyle\sum_{\substack{n > z \\ P^+(s) \le y}} \frac{1}{s} A_y\left(\frac{2}{h(s)}\right) \le \left(\prod_{p \le y}\left(1 - \frac{1}{p}\right)\right) \sum_{\substack{s > z \\ P^+(s) \le y}} \frac{1}{s}$

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Bound the tail sum trivially: $\displaystyle\sum_{\substack{n>z\\ P^+(s)\le y}} \frac{1}{s} A_y\left(\frac{2}{h(s)}\right) \le \left(\prod_{p\le y}\left(1-\frac{1}{p}\right)\right) \sum_{\substack{s>z\\ P^+(s)\le y}} \frac{1}{s}$

Bound the $A_y(x)$ in the first sum using moments of $h_y(n) := \sum_{\substack{d|n\\ P^-(d)>y}} \frac{1}{d}$.

Set $\mu_{y,r} := \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} h_y(n)^r$ then $A_y(x) \le \prod_{p\le y}\left(1-\frac{1}{p}\right) \frac{\mu_{y,r}-1}{x^r-1}$

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Deléglise takes $y = 500$, $z = 10^{14}$ to get $d(\mathcal{A}) < 0.2480$.

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The integer n is a covering number $\iff r(n) = n \iff c(n) = 2$.

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$$d(\mathcal{C}) < \sum_{\substack{s \leq z \\ P^+(s) \leq y}} \frac{1}{s} A_y\left(\frac{2}{c(s)}\right) + \left(\prod_{p \leq y}\left(1 - \frac{1}{p}\right)\right) \sum_{\substack{s > z \\ P^+(s) \leq y}} \frac{1}{s}.$$

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Unfortunately, computing $c(n)$ is **very** computationally intensive...

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Theorem

If $n = \ell m$ with $(\ell, m) = 1$ and $r(\ell) = \ell - 1$ then

$$c(n) \leq 1 + \frac{\ell - 1}{\ell} + \frac{1}{\ell} \sum_{d|m,\, d>1} \frac{B_{\tau(\ell),\omega(d)}}{d} =: c'(n).$$

Using c'(n) to bound the density

$$d(\mathcal{C}) < \sum_{\substack{s \le z \\ P^+(s) \le y}} \frac{1}{s} A_y\left(\frac{2}{c'(s)}\right) + \left(\prod_{p \le y}\left(1 - \frac{1}{p}\right)\right) \sum_{\substack{s > z \\ P^+(s) \le y}} \frac{1}{s}$$

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No hope of computing this sum with $y = 500$, $z = 10^{14}$ like Deléglise.

With $y = 200$, $z = 10^9$ we found:

$$0.1032 \le d(\mathcal{C}) \le 0.1197$$

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Smooth-Rough Divisor Partition

Definition

For a pair (a, q) with $P^+(a) \le q$, denote by $M_{a,q} = \{ar : P^-(r) \ge q\}$.

A set W of such pairs is a **smooth-rough divisor partition** of $\mathbb{N}$ if:

- $M_{a,q}$ are disjoint,
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Further partition $W = W_< \sqcup W_=$

$$W_< := \{(a, q) \in W : P^+(a) < q\}$$
$$W_= := \{(a, q) \in W : P^+(a) = q\}$$

Smooth-Rough-Divisor Partitions

Example:

$$W = \{(1,5),(2,3),(3,3),(4,2)\}$$

$$W_< = \{(1,5),(2,3)\},$$
$$W_= = \{(3,3),(4,2)\}$$

For any smooth-rough divisor partition W,

$$d(\mathcal{C}) = \sum_{(a,q)\in W_<} \frac{1}{a} A_q\left(\frac{2}{c'(a)}\right) + \sum_{(a,q)\in W_=} \frac{1}{a(1-1/q)} A_{q+1}\left(\frac{2}{c''(a,q)}\right)$$

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After much numerical experimentation and several hours of computation...

Theorem

$$0.10324 < d(\mathcal{C}) < 0.10356$$

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Identify all primitive covering numbers $< 773\,500 = 2^2 \times 5^3 \times 7 \times 13 \times 17$.

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Recall, Kobayashi (2010): $0.2476171 < d(\mathcal{A}) < 0.2476475$.

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Can we use them to bound $d(\mathcal{A})$? Yes!

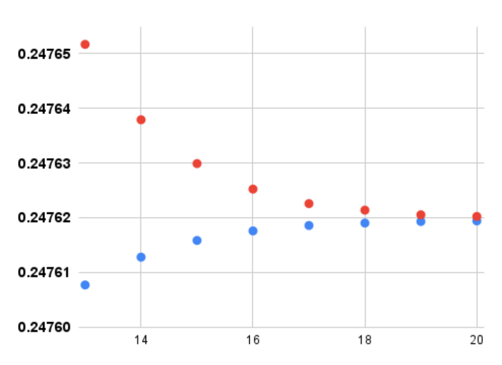
Recall, Kobayashi (2010): $0.2476171 < d(\mathcal{A}) < 0.2476475$.

Using smooth-rough divisor partitions and 3 minutes of runtime we find

$$0.2476127 < d(\mathcal{A}) < 0.2476379.$$

Abundant density bounds

Lower	Upper	Time
0.2476127	0.2476379	3m07s
0.2476158	0.2476299	13m08s
0.2476175	0.2476253	56m56s
0.2476185	0.2476226	4h54m59s
0.2476190	0.2476214	26h38m12s
0.24761929	0.24762053	6d05h14m31s
0.24761940	0.24762022	



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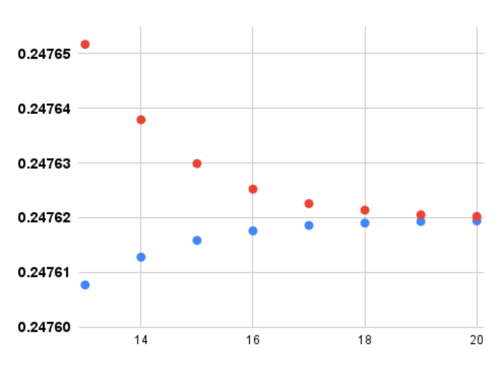
0.2476190	0.2476214	26h38m12s
0.24761929	0.24762053	6d05h14m31s
0.24761940	0.24762022	



Further optimization and about two weeks of distributed computation gives:

Theorem

$$0.24761951 < d(\mathcal{A}) < 0.24761989$$

THANK YOU!