Counting primitive subsets of $\{1, 2, \ldots n\}$

Nathan McNew
Towson University

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How many primitive subsets of $\{1, 2 \ldots n\}$ are there?

Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most n.

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A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

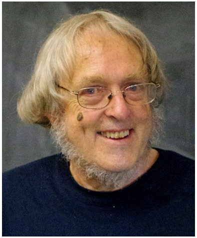
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

Bounds

Cameron and Erdős considered the function $Q(n)$ in 1990.



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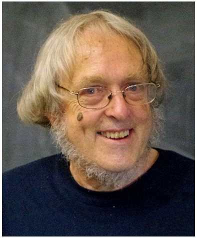

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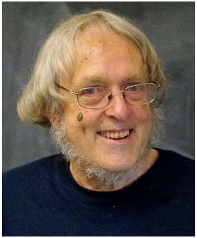

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Every subset of integers contained in $\{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, n\}$ is primitive, and there are $2^{\lceil \frac{n}{2} \rceil} > \sqrt{2}^n$ such subsets.

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Conjecture: $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

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Proof is by considering sets in which the ratio between any two terms is not a k-smooth integer. Then let $k \to \infty$.

Say n is k-smooth if the largest prime divisor $P^+(n) \leq k$.

Proof gives no insight to the value of the limit nor improved bounds.

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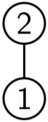
This is the same as counting primitive subsets of $\{1, 2\}$ or independent subsets of the graph:



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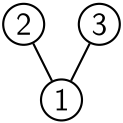
If k odd this is equivalent to counting primitive subsets of $\{1,2,3\}$, which is 5, replacing the 4 possibilities before.



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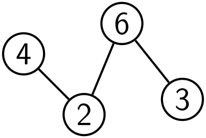

8 possibilities $\{\}$,$\{2\}$,$\{3\}$,$\{4\}$,$\{6\}$,$\{2,3\}$,$\{3,4\}$,$\{4,6\}$ replace 6 before.

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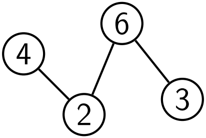

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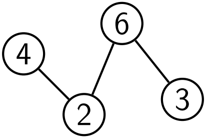

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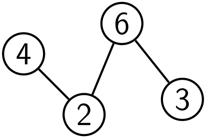

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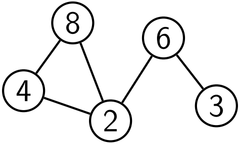
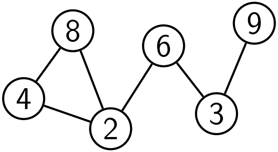


However, if $k \leq \frac{2n}{9}$ then $9\ell \leq n$ must also be considered.



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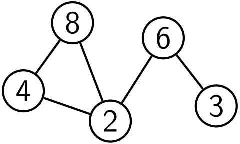

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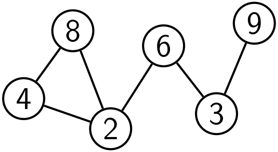

So we consider $2\ell, 3\ell, 4\ell, 6\ell, 8\ell$ and 9ℓ.

Key observation

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$ be the contribution of k to our potential primitive set.

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Let $i = \left\lfloor \frac{n}{k} \right\rfloor$, so the multiples of k up to n are k, $2k$, $\ldots ik$. (All included in connected component of k.)

None of the ratios in this component can involve a prime greater than or equal to i (except possibly the edge from k to ik if i is prime.)

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Then $r(i,n) = r(d\ell, t\ell) = r(d,t)$.

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We have

$$Q(n) = \prod_{k=1}^{n} r(k,n)$$
$$= c^n \left(\prod_{i=1}^{N} \prod_{\substack{d \\ P^+(d)<i \\ d<M}} \prod_{t\in[id,(i+1)d)} r(d,t)^{O(1)} \right)$$
$$\times\, exp \left(O\left(\frac{n}{N} + \sum_{i=1}^{N} \#\{k \le n : d|k, d > M, P^+(d) < i\} \right) \right)$$

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Take $M = n^{1/2-\epsilon}$, $N = \exp\left((\log n)^{1/2-\epsilon}\right)$.

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~~#24418~~ closed defect (fixed)

8.2 β4

Opened 5 months ago
Closed 4 months ago

Doctest: bug numerical_approx(2^(450232897/4888643760))

Reported by:	vdelecroix	Owned by:	
Priority:	major	Milestone:	~~sage-8.2~~
Component:	symbolics	Keywords:	bug
Cc:	rws	Merged in:	
Authors:	Ralf Stephan	Reviewers:	Jeroen Demeyer
Report Upstream:	N/A	Work issues:	
Branch:	821f7d9 (Commits)	Commit:	821f7d9f3568316bc0b8b1f5619bce...
Dependencies:		Stopgaps:	

Description (last modified by vdelecroix) Δ

```
sage: numerical_approx(2^(450232897/4888643760))
---------------------------------------------------------------------------
RuntimeError                              Traceback (most recent call last)
<ipython-input-2-3c4e30ac02c1> in <module>()
----> 1 numerical_approx(Integer(2)**(Integer(450232897)/Integer(4888643760)))

/opt/sage/local/lib/python2.7/site-packages/sage/misc/functional.pyc in numerical_ap
   1406         return numerical_approx_generic(x, prec)
   1407     else:
```

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As before,

$$Q(n) = \prod_{k=1}^{n} (g(k) + 1)$$

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Note: $g(p) = \frac{Q(p-1)-1}{Q(p-1)}$.

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This gives (conjecturally)

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Thank you!