Primitive and Geometric Progression Free Sets Without Large Gaps

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May 23rd, 2017

Gaps between prime numbers

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In theory: At most 10^{53}. (Best we can prove)

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Conjecture (Cramér)

$$p_n - p_{n-1} \le \log^2 p_n$$

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Summary: The largest gap between primes up to x has size between

$$\frac{\log x \log\log x \log\log\log\log x}{\log\log\log x} \quad \text{and} \quad x^{0.525}$$

but we think the truth is $\log^2 x$.

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Definition

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  For example, $\mathcal{P}_2 = \{4, 6, 9, 10, 14, 15, 21, 22, \ldots\}$.

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Theorem (Erdős)

If S is a primitive set then

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Conjecture (Erdős)

If S is a primitive set then

$$\sum_{n \in S} \frac{1}{n \log n} < \sum_{p \in \mathcal{P}} \frac{1}{p \log p} = 1.63\ldots.$$

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Due to the nature of the construction of these sets, it isn't possible to get good upper bounds for the largest gap in such a set.

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$$a_n - a_{n-1} \leq \exp\left(\sqrt{(2+\epsilon)\log a_n \log\log a_n}\right).$$

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There exists a positive constant c such that every interval $[x - cy, x]$ contains at least $\frac{y}{\log^2 y}$ integers free of prime factors smaller than $\sqrt{y}$.

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$$P(X_i = m) = \frac{C_\epsilon}{m^{1+\epsilon/4}}$$

with $C_\epsilon = \frac{1}{\zeta(1+\epsilon/4)}$ chosen to normalize the distribution.

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Note Q is primitive: If $a \in Q$, and $a|b$ with $b > a$, then $\Omega(a) < \Omega(b)$, but any prime dividing a also divides b, and so b cannot be in Q.

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$$\geq \left(\frac{C_\epsilon}{\left(\frac{2\log x}{\log y}\right)^{1+\epsilon/4}}\right)^{\frac{2\log x}{\log y}} = \exp\left(-(2+\tfrac{\epsilon}{2}+o(1))\frac{\log x}{\log y}\log\left(\frac{\log x}{\log y}\right)\right)$$

as $x \to \infty$.

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The probabilistic method

Because the elements of S are pairwise coprime, their probabilities of being included in Q are independent.
Thus the probability that no element of S is included in Q is

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&\quad\le \exp\left(-\frac{c'\sqrt{y}}{\log y}\times\exp\left(-(2+\tfrac{\epsilon}{2}+o(1))\frac{\log x}{\log y}\log\left(\frac{\log x}{\log y}\right)\right)\right)\\
&\quad= \exp\left(-\exp\left(\tfrac{1}{2}\log y-(2+\tfrac{\epsilon}{2}+o(1))\frac{\log x}{\log y}\log\left(\frac{\log x}{\log y}\right)\right)\right).
\end{aligned}$$

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$$\prod_{n \in S} (1 - P(n \in Q)) \\ \leq \exp\left(-\exp\left(\tfrac{1}{2}\log y - (2 + \tfrac{\epsilon}{2} + o(1))\frac{\log x}{\log y}\log\left(\frac{\log x}{\log y}\right)\right)\right).$$

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$$= \frac{\epsilon}{4\sqrt{2+\epsilon}}\sqrt{\log x \log\log x} = C'_\epsilon\sqrt{\log x \log\log x}$$

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The probability that no integer in $[x - cy, x]$ is included in Q is at most $\exp\left(-\exp\left(C'_\epsilon\sqrt{\log x \log\log x}\right)\right)$.

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Thus there exists a set, Q satisfying the properties of the theorem.

Pairwise coprime sets

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Theorem (M.)

For every $\epsilon > 0$ there exists a pairwise coprime set $b_1, b_2, b_3 \ldots$, such that

$$b_n - b_{n-1} \ll b_n^{\alpha+\epsilon}$$

where $\alpha = \frac{5-\sqrt{17}}{2} = 0.43845\ldots$

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The squarefree integers $s_1, s_2, \ldots$ satisfy

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We can find larger sets that avoid progressions, but can't say much about the gaps of these sets.

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Theorem (He, 2015)

For each $\epsilon > 0$ *there exists a set* $\{t_1, t_2, \ldots\}$ *avoiding 6-term geometric progressions (rational ratio) with gaps satisfying*

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Note V avoids 3-term progressions: If (a, ar, ar^2) is a progression, a prime in r appears to different powers in at least two of these terms.

Thank you!