AVOIDING 3-TERM GEOMETRIC PROGRESSIONS IN NON-COMMUTATIVE SETTINGS

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ABSTRACT. Several recent papers have considered the Ramsey-theoretic problem of how large a subset of integers can be without containing any 3-term geometric progressions. This problem has also recently been generalized to number fields and $\mathbb{F}_q[x]$. We study the analogous problem in two noncommutative settings, quaternions and free groups, to see how lack of commutivity affected the problem. In the quaternion case, we show bounds for the supremum of upper densities of 3-term geometric progression avoiding sets. In the free groups case, we calculate the decay rate for the greedy set in $\langle x, y : x^2 = y^2 = 1\rangle$ avoiding 3-term geometric progressions.

CONTENTS

1. INTRODUCTION

Classically, there has been interest in how large a set can be while still avoiding arithmetic or geometric progressions. In a 1961 paper Rankin [Ran] introduced the idea of considering how large a set of integers can be without containing terms which are in geometric progression. He constructed a subset of the integers which avoids 3-term geometric progressions and has asymptotic density approximately 0.719745. Brown and Gordon [BG] noted that the set Rankin considered was the set obtained by greedily including integers

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subject to the condition that such integers do not create a progression involving integers already included in the set.

This question has been generalized to number fields [BHMMPTW] and polynomial rings over finite fields [AFGMMMM]. The purpose of [BHMMPTW] was to see how changing from subsets of $\mathbb{Z}$ to subsets of number fields affected the answer, while in [AFGMMMM] it was to see how the extra combinatorial structure of $\mathbb{F}_q[x]$ affected the tractability and features of the problem. In our case, we wish to see how non-commutativity affects the answer.

The first half of this paper (Sections 2 through 5) is dedicated to studying the problem in the Hurwitz order quaternions, Q_{Hur} (see Section 2 for a review of their properties). We consider sets avoiding geometric progression of the form a, ar, ar^2 with $a, r \in Q_{\text{Hur}}$, being careful to specify the order of multiplication due to the non-commutativity of the algebra. We produce some bounds on the supremum of upper densities of sets avoiding 3-term geometric progressions, and use Rankin's greedy set to construct a similar set avoiding 3-term geometric progressions in the Hurwitz order quaternions. We also discuss the peculiarities of this setting in Section 5. The second half (Section 6) is dedicated to studying the question in the setting of free groups. We arrive at the following results.

Theorem 3.1. *Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then*

$$.946589 \;\leq\; m_{\text{Hur}} \;\leq\; .952381. \tag{1.1}$$

Theorem 4.2. *Let Q_{Ran} be the set of Hurwitz quaternions with norm in Rankin's greedy set (avoiding 3-term geometric progressions in $\mathbb{Z}$). Let $A_3^*(\mathbb{Z})$ be the greedy set avoiding 3-term arithmetic progressions. The asymptotic density of Q_{Ran} is*

$$d(Q_{\text{Ran}}) \;=\; \left(\prod_{p \text{ odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}\right]\right) \cdot \left(\sum_{n \in A_3^*(\mathbb{Z})} \frac{2^2 - 1}{2^2 2^{2n}}\right) \approx 0.771245. \tag{1.2}$$

Theorem 6.2. *Let $\mathcal{G} = \langle x, y : x^2 = y^2 = 1\rangle$ be the free group on two generators each of order two. Order the group as $W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \ldots)$ and take the set G formed by greedily taking elements that don't form a 3-term progression with previously added ones. Then*

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq 2 \cdot 3^n\}|}{|\{w \in W : \text{length}(w) \leq 2 \cdot 3^n|\}|} \;=\; \frac{2^{n+1}}{1 + 4 \cdot 3^n}, \tag{1.3}$$

and in general

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq n\}|}{|\{w \in W : \text{length}(w) \leq n\}|} \;=\; \Theta\left((2/3)^{\log_3 n}\right). \tag{1.4}$$

2. Review of Hurwitz order quaterions

When considering a non-commutative analogue of the geometric-progression-free set problem, the quaternions are a natural choice to consider first, as they form a non-commutative algebra and have a norm. The Hamiltonian quaternions can be subdivided into two orders with integral properties: the Lipschitz and the Hurwitz orders. We will restrict our attention to the Hurwitz order of quaternions, due to the existence of prime factorization in the Hurwitz order.

Definition 2.1. *Quaternions constitute the algebra over the reals generated by units i, j, and k such that*

$$i^2 \;=\; j^2 \;=\; k^2 \;=\; ijk \;=\; -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

Definition 2.2. *We say that $a + bi + cj + dk$ is in the Hurwitz order, Q_{Hur}, if a, b, c, d are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$.*

Definition 2.3. *The norm of a quaternion $Q = a + bi + cj + dk$ is given by* $\text{Norm}(Q) = a^2 + b^2 + c^2 + d^2$.

An element $P \in Q_{\text{Hur}}$ is said to be prime if and only if its norm is prime.

Theorem 2.4. *Let $Q \in Q_{\text{Hur}}$. For every factorization* $\operatorname{Norm}(Q) = p_0 p_1 \cdots p_k$ *of the norm, there is a factorization*

$$Q \;=\; P_0 \cdots P_k \tag{2.1}$$

of Q into Hurwitz primes such that such that $N(P_i) = p_i$ for all $0 \leq i \leq k$. We call such a factorization modelled on the factorization $p_0 \ldots p_k$ of $\operatorname{Norm}(Q)$*. Furthermore, any other factorization modelled on* $\operatorname{Norm}(Q) = p_0 \cdots p_k$ *is of the form:*

$$Q \;=\; P_0 U_1 \cdot U_1^{-1} P_1 U_2 \cdots U_k^{-1} P_k. \tag{2.2}$$

That is, the factorization is unique up to unit-migration, also known in this setting as metacommutation.

For a proof see Chapter 5.2, Theorem 2 in [ConSm].

We need a few facts about Hurwitz order quaternions to calculate some of the densities and bounds. Namely, we want to know the number of Hurwitz quaternions up to a certain norm, the number of Hurwitz quaternions of a particular norm, and the proportion of Hurwitz quaternions whose norm is divisible exactly by p^n. Readers with knowledge of the Hurwitz quaternions may wish to skip Section 2.1 and briefly skim Section 2.2. Section 2.1 will be used throughout, and Section 2.2 will specifically be useful for Section 4. For a more in-depth discussion of the Hurwitz order, see Chapter 5 in [ConSm].

2.1. **Counting quaternions up to a given norm.** We wish to count the number of Hurwitz quaternions with norm in $[0, M]$. In order to do this, we need the number of quaternions of a specific norm.

Lemma 2.5. *The number of Hurwitz quaternions of norm N is*

$$S(\{N\}) \;=\; 24 \sum_{2 \nmid d \mid N} d, \tag{2.3}$$

the sum of the odd divisors of N multiplied by 24.

See [ConSm] for a proof. This fact allows us to prove the following lemma.

Lemma 2.6. *The number of Hurwitz quaternions with norm less than or equal to M is*

$$|S(M)| \;=\; \pi^2 M^2 + O(M \log M) \tag{2.4}$$

Proof. The number of Hurwitz quaternions up to some norm M is

$$\begin{aligned}
S(M) \;&=\; 24 \sum_{n \leq M} \sum_{\substack{d \mid n \\ 2 \nmid d}} d \\
&=\; 24 \sum_{d \leq M} \sum_{\substack{e \leq \lfloor \frac{M}{d} \rfloor \\ 2 \nmid e}} e \\
&=\; 24 \sum_{d \leq M} \left\lceil \frac{\lfloor \frac{m}{d} \rfloor}{2} \right\rceil^2 \\
&=\; \frac{24}{4} \sum_{d \leq M} \left(\left(\frac{M}{d} \right)^2 + O\left(\frac{M}{d} \right) \right) \\
&=\; 6M^2 \sum_{d \leq M} \frac{1}{d^2} + O\left(\sum_{d \leq M} \frac{M}{d} \right) \\
&=\; 6M^2 \left(\frac{\pi^2}{6} \right) + O\left(M^2 \sum_{d > M} \frac{1}{d^2} + \sum_{d \leq M} \frac{M}{d} \right) \\
&=\; \pi^2 M^2 + O\left(M \log M \right).
\end{aligned} \tag{2.5}$$

Note that in the third line, we used the fact that the sum of the first n odd numbers is n^2. $\square$

2.2. **Prime divisors of Quaternion norms.**

Lemma 2.7. *If p is odd, the proportion of Hurwitz quaternions whose norm is divisible by p^n but not p^{n+1} is*

$$\frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}. \tag{2.6}$$

If $p=2$, this proportion is instead

$$\frac{2^2-1}{2^2 2^{2n}} = \frac{3}{4\cdot 2^{2n}}. \tag{2.7}$$

Proof. We first calculate the proportion of Hurwitz quaternions whose norm is divisible by p^n but not by p^{n+1}. Consider the set $S(N)$, the set of Hurwitz quaternions with norm greater than or equal to N. Since we can always find a factorization of h based off any permutation of the prime factors of $N(h)$ (Theorem 2.4), we can always write

$$h = PH \tag{2.8}$$

where $N(P)=p^k$, with k being the largest power of p that divides $N(h)$. There are 24 ways to write h in this form, since $h = Pu\cdot u^{-1}H$ as well. Thus the proportion of elements of $S(N)$ that have at least a factor of p^n in their norm is

$$\frac{|S(\{p^n\})|\cdot|S(N/p^n)|}{24|S(N)|}. \tag{2.9}$$

From Lemma 2.5 we calculate that for an odd prime p,

$$S(\{p^n\}) = 1+p+p^2+\cdots+p^n = \frac{p^{n+1}-1}{p-1}. \tag{2.10}$$

Note that $|S(N)| = \pi^2N^2+O(N\log N)$ by Lemma 2.6 . Substituting all this information into Equation (2.9) yields

$$\frac{(p^{n+1}-1)(\pi^2(N/p^n)^2+O(N\log N))}{(p-1)(\pi^2N^2+O(N\log N))}. \tag{2.11}$$

Subtracting the proportion of $S(N)$ of elements whose norm is at least divisible by p^{n+1} and taking the limit as $N\to\infty$, we get the proportion of elements of $S(N)$ whose norm is divisible by p^n but not divisible by p^{n+1}.

$$\frac{(p^{n+1}-1)(\pi^2(N/p^n)^2+O(N\log N))}{(p-1)(\pi^2N^2+O(N\log N))} - \frac{(p^{n+2}-1)(\pi^2(N/p^{n+1})^2+O(N\log N))}{(p-1)(\pi^2N^2+O(N\log N))}. \tag{2.12}$$

Taking the limit $N\to\infty$ gives the proportion of Q_{Hur} whose norm is exactly divisible by p^n.

$$\lim_{n\to\infty}\frac{(p^{n+1}-1)(\pi^2(N/p^n)^2)-(p^{n+2}-1)(\pi^2(N/p^{n+1})^2)+O(N\log N)}{(p-1)(\pi^2N^2)+O(N\log N)} = \frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}. \tag{2.13}$$

An analogous calculation for $p=2$, using that $S(\{2^k\}) = \sum_{2\nmid d|2^k} d = 1$, gives us that the proportion of elements whose norm is exactly divisibly by 2^n is

$$\frac{2^2-1}{2^2 2^{2n}} = \frac{3}{4\cdot 2^{2n}}. \tag{2.14}$$

□

3. Bounds on the supremum of the upper densities

3.1. **Lower bound.** Viewing the Hurwitz quaternions embedded into $\mathbb{R}^4$, we use the fact that the norm of the smallest non-unit Hurwitz quaternion is 2 to construct a union of six hyperspheric annuli that does not contain any geometric progressions. Note that we are restricting our geometric progressions to those formed by multiplication on the right by a constant ratio in the Hurwitz order. We choose the norm ranges, i.e., unions of intervals in $\mathbb{R}_{\geq 0}$, that induce these annuli to avoid geometric progressions in the norm, which implies we avoid geometric progressions in the quaternion elements themselves.

This construction is done in [McN] for the integers. Since every integer is realized as the norm of a Hurwitz quaternion, the intervals chosen there also work in our case.

For $S(M)$, M large, consider $S((M/4, M])$. Since the smallest non-unit ratio for a geometric progression is 2, this set has no 3-term progressions in the norms, and thus cannot have any 3-term progressions in its elements. Thus, the proportion of elements in $S((M/4, M])$ compared to $S(M)$ is

$$\frac{|S((M/4,M])|}{|S(M)|} = \frac{(1/8)V(2\sqrt{M}) - (1/8)V(2\sqrt{M/4}) + O(M\log M)}{(1/8)V(2\sqrt{M}) + O(M\log M)} = \frac{\pi^2 M^2 - \pi^2 (M/4)^2 + O(M\log M)}{\pi^2 M^2 + O(M\log M)}. \tag{3.1}$$

As $M \to \infty$ this proportion goes to $1 - 1/16 = 15/16$. We can get a higher proportion by including more annuli. We define

$$T_M := \left(\frac{M}{48}, \frac{M}{45}\right] \cup \left(\frac{M}{40}, \frac{M}{36}\right] \cup \left(\frac{M}{32}, \frac{M}{27}\right] \cup \left(\frac{M}{24}, \frac{M}{12}\right] \cup \left(\frac{M}{9}, \frac{M}{8}\right] \cup \left(\frac{M}{4}, M\right]. \tag{3.2}$$

Fix $N = 1$ and $N_i = 48^2 N_{i-1}^2$ for $i \geq 2$. Consider

$$S_N = \bigcup_{M \in \mathbb{N}} T_M. \tag{3.3}$$

The proof that S_N avoids geometric progressions can be found in Theorem 3.1 of [McN]. The upper density of this set is the proportion $|T_M|/|S(M)|$ as $M \to \infty$:

$$\begin{aligned}
\overline{d}(S_N) &= \lim_{M\to\infty} \frac{|T_M|}{|S(M)|} \\
&= \lim_{M\to\infty} \frac{1}{|S(M)|} \cdot \Big[S((M/48, M/45]) + S((M/40, M/36]) + S((M/32, M/27]) + \\
&\qquad\qquad S((M/24, M/12]) + S((M/9, M/8]) + |S((M/4, M])| \Big] \\
&= \frac{1}{M^2} \Big[[M^2 - (M/4)^2] + [(M/8)^2 - (M/9)^2] + [(M/12)^2 - (M/24)^2] + \\
&\qquad [(M/27)^2 - (M/32)^2] + [(M/36)^2 - (M/40)^2] + [(M/45)^2 - (M/48)^2] \Big] \\
&= \Big[[1 - 1/4^2] + [1/8^2 - 1/9^2] + [1/12^2 - 1/24^2] + \\
&\qquad [1/27^2 - 1/32^2] + [1/36^2 - 1/40^2] + [1/45^2 - 1/48^2] \Big],
\end{aligned} \tag{3.4}$$

which to six decimal places is .946589.

3.2. **Upper bound.** We generalize a construction done in [McN] where we show a certain proportion of elements are forced to be removed to avoid three-term progressions. Namely, we look at disjoint 3-tuples $(b, rb, r^2 b)$, from which one element must be excluded. We pick r to have the smallest norm, 2, to get a large number of exclusions.

By Lemma 2.5, there is one prime r of norm 2 up to unit multiples on either side. As an analogue of "coprime", by Lemma 2.7, three-fourths of Hurwitz quaternions have no power of 2 in their norm, and thus contain no factors of r in their factorization.

Fix r a prime of norm 2. Consider $S(M)$ for large M. Then if $b \in S(M)$, $N(b) \leq M/4$, and b has no power of 2 in its norm, then b, rb, rb^2 forms a progression, and all such sequences are disjoint for different b.

If b has no power of 2 in its norm and $N(b) \leq M/2^5$, then a similar argument follows with $r^3 b, r^4 b, r^5 b$. Looking at the norms of $r^3 b, r^4 b, r^5 b$, these sequences are disjoint from the $b', rb', r^2 b'$ sequences from before. So for each b (up to units on the left) with no power of 2 in its norm, and $N(b) \leq M/2^5$, we need to make an additional exclusion to avoid three-term progressions. Taking $M \to \infty$ we get an upper bound of

$$\begin{aligned}
\lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{|S(M/2^2)| + |S(M/2^5)|}{|S(M)|}\right) &= \lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{\pi^2 M^2/2^4 + \pi^2 M^2/2^{10} + O(M\log M)}{\pi^2 M2 + O(M\log M)}\right) \\
&= 1 - \frac{3}{2^6} - \frac{3}{2^{12}} \\
&\approx .952393.
\end{aligned} \tag{3.5}$$

We can improve this bound slightly by considering more b, though just taking the b above is already quite close to the truth. Looking at b's in $S(M/2^2), S(M/2^5), S(M/2^8), \dots$ we get an upper bound of

$$\begin{aligned}
\lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{|S(M/2^2)| + |S(M/2^5)| + \cdots + |S(M/2^{2+3i})| + \cdots}{|S(M)|}\right) &= 1 - \frac{3}{4}\cdot\frac{1}{2^4}\cdot\sum_{i=0}^{\infty}\frac{1}{2^{6i}} \\
&= 1 - \frac{3}{2^6}\left(\frac{1}{1-1/2^6}\right) \\
&= 1 - \frac{3}{2^6-1} \\
&\approx .952381. \qquad (3.6)
\end{aligned}$$

From the two subsections, we get the following.

Theorem 3.1. *Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then*

$$.946589 \leq m_{\mathrm{Hur}} \leq .952381. \qquad (3.7)$$

4. Density of Rankin's quaternion greedy set

Consider the set $G_3^*(\mathbb{Z}) = \{1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, \dots\}$, which we refer to as Rankin's (geometric) greedy set; $G_3^*(\mathbb{Z})$ is the set formed by greedily including integers that do not form 3-term geometric progressions with the previous elements. Since geometric progressions give arithmetic progressions in their terms' prime powers, $G_3^*(\mathbb{Z})$ is the set of elements whose prime factors' exponents are all in $A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 8, 10, 12, 13 \dots\}$, the set formed by greedily taking integers that do not form an arithmetic progression. Let $A_3^*(\mathbb{Z})$ be the set of integers whose ternary expansion do not contain the digit 2.

Definition 4.1. *We define Q_{Ran} as the set of Hurwitz quaternions whose norm is in Rankin's greedy set:*

$$Q_{\mathrm{Ran}} := \{h \in Q_{\mathrm{Hur}} : N(h) \in G_3^*(\mathbb{Z})\}. \qquad (4.1)$$

Since this set avoids progressions in the norms of its elements, it avoids progressions in its quaternion elements. We wish to deduce the density of this set. We do this be calculating the probably that an element has norm divisible by a suitable power of p.

Theorem 4.2. *The asymptotic density of Q_{Ran} is*

$$d(Q_{\mathrm{Ran}}) = \left(\prod_{p \text{ odd}}\left[\sum_{n\in A_3^*(\mathbb{Z})}\frac{p^{n+3}-p^{n+2}-p^2+1}{p^2(p-1)p^{2n}}\right]\right)\cdot\left(\sum_{n\in A_3^*(\mathbb{Z})}\frac{2^2-1}{2^2 2^{2n}}\right). \qquad (4.2)$$

Proof. By Lemma 2.7 the probability that the norm of a Hurwitz quaternion has norm exactly divisible by p^n, for p odd, is

$$\frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}, \qquad (2.6)$$

and $(2^2-1)/(2^2 2^{2n})$ for $p = 2$. So the probability that the norm of a Hurwitz quaternion has a proper power of an odd p (that is, a power of p in $A_3^*(\mathbb{Z})$) is

$$\sum_{n\in A_3^*(\mathbb{Z})}\frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}. \qquad (4.3)$$

Note that in Equation (2.12), with respect to the p factors the expression is $\sim 1/p^n$, so even with the error terms we should get proper convergence of the sum. The proportion of Hurwitz quaternions with a proper power of 2 in their norm is

$$\left(\sum_{n\in A_3^*(\mathbb{Z})}\frac{2^2-1}{2^2 2^{2n}}\right). \qquad (4.4)$$

By a Chinese Remainder Theorem-type argument, we get the desired product:

$$d(Q_{\mathrm{Ran}}) = \left(\prod_{p \text{ odd}} \left[\sum_{n\in A_3^*(\mathbb{Z})} \frac{p^{n+3}-p^{n+2}-p^2+1}{p^2(p-1)p^{2n}}\right]\right)\cdot\left(\sum_{n\in A_3^*(\mathbb{Z})}\frac{2^2-1}{2^2 2^{2n}}\right). \tag{4.2}$$

□

This sum is slowly converging and estimated at 0.771245 through computational methods. This is slightly higher than Rankin's density on the integers at about 0.719745 [Ran] due to the variation in the number of quaternions per norm that are added to Q_{Ran} for each element of $G_3^*(\mathbb{Z})$.

5. The quaternion greedy set

In a similar style to Rankin's argument, we can form a greedy set of quaternions, which we call $G_3^*(\mathrm{Hur})$, by including quaternions of increasing norm so long as they do not form a geometric progression with elements of smaller norm already included in the set. This process begins with including all the unit Hurwitz quaternions of norm 1 and then considers progressively larger norms. This set will be well defined since including a particular quaternion of a given norm, n, will not create a geometric progression with any other quaternions of norm n since unit ratios are not allowed. Therefore, the greedy set will be the same regardless of the order in which quaternions of a given norm are added.

This greedy construction creates a set similar to Q_{Ran}. However, the properties of the quaternions result in behavior that is substantially more complicated than Q_{Ran}. For example there exist quaternions of norm 49 that cannot be written as the square of a quaternion of norm 7 multiplied by a unit on the left. For example, the quaternion 7 cannot be so represented.

Proposition 5.1. *The hurwitz quaternion 7, despite having a norm, 49 which is a perfect square, cannot be represented in the form $7 = UR^2$, where U is a unit.*

Proof. Suppose for contradiction that 7 can be written $7 = UR^2$ as the square of a Hurwitz quaternion R of norm 7 multiplied by a unit U on the left. Then $7U^{-1} = R^2$. Explicitly, let $U^{-1} = (a + bi + cj + dk)$, also a hurwitz quaternion and $R = (e + fi + gj + hk)$. R has norm 7 and 7 cannot be written as the sum of three squares, e through h must be nonzero, and given the restriction on the norm of R we must have $e, f, g, h \in \{\pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}, \pm 1, \pm 2\}$. Multiplying out $R^2 = (e^2 - f^2 - g^2 - h^2) + 2efi + 2egj + 2ehk$, equating with $7U^{-1} = 7a + 7bi + 7cj + 7dk$, and solving for a, b, c, d we find that $a = \frac{e^2-f^2-g^2-h^2}{7}$, $b = \frac{2ef}{7}$, $c = \frac{2eg}{7}$, and $d = \frac{2eh}{7}$. Since e through h are nonzero, $a, b, c, d \in \{\pm\frac{1}{2}\}$. Thus $|ef| = \frac{7}{4}$. This leads to a contradiction due to the limitations on the values for e and f. □

One can similarly find that all of the quaternions in $\{\pm 7, \pm 7i, \pm 7j, \pm 7k\}$ cannot be so represented. So these Hurwitz quaternions of norm 49 will not be part of any geometric progression involving the units and the quaternions of norm 7. This results in some elements of norm 49 being included in $G_3^*(\mathrm{Hur})$, whereas 49 is not in $G_3^*(\mathbb{Z})$ and therefore no elements of norm 49 are contained in Q_{Ran}. As a result, some items of norm 343 form a geometric progression in $G_3^*(\mathrm{Hur})$ and thus are excluded while all Hurwitz quaternions of norm 343 are included in Q_{Ran}. This sequence of inclusions and exclusions continues for all powers of 7.

This behavior is not unique to the quaternions of norm 7 and in fact occurs for all integers that cannot be written as the sum of three squares. Furthermore, any integer with an odd divisor greater than 23 poses the same problem.

Lemma 5.2. *If n is divisible by an odd integer greater than 23, then there exists a Hurwitz quaternion Q of norm n^2 which cannot be written in the form $Q = UR^2$, where U is a unit and R is a Hurwitz quaternion of norm n. Hence Q is not part of any 3 term geometric progression of the form U, UR, UR^2.*

Proof. Lemma 2.5 allows us to write $S(\{n\})$ as $24\sum_{2\nmid d|n} d$ and $S(\{n^2\})$ as $24\sum_{2\nmid d|n^2} d$. Then the number of possibilities for a square of norm n multiplied by a unit on the left is $24 * S(\{n\})$. The proof will be complete if we can show that

$$24 * S(\{n\}) < S(\{n^2\}).$$

Let D be the greatest odd divisor of n. Then we have

$$24 * D * S(n) = 24 * D * \sum_{2\nmid d|n} d \leq 24 * \sum_{2\nmid d|n^2} d = 24 * S(\{n^2\}).$$

Thus, if $D > 23$, then

$$24 * S(\{n\}) \ < \ D * S(\{n\}) \leq S(\{n^2\}).$$

Therefore by a simple counting argument, the set of quaternions with norm n, where n has an odd divisor greater than 23, cannot square to realize all quaternions of norm n^2. □

In practice, the greedy set of Hurwitz quaternions results from a large number of these inclusions and exclusions which so far appear to be hard to predict or keep track of. As a result we do not know whether the density of $G_3^*(\mathrm{Hur})$ is greater or less than the density of Q_{Ran}. Furthermore, the large number of Hurwitz quaternions and the nature of these inclusions and exclusions has made a computational estimate of the density of this greedy set difficult.

6. Free groups on two generators of order two

6.1. **Introduction.** We now consider the case of subsets of free groups containing no three-term geometric progressions. Due to the nature of free groups and not being able to space out geometric progressions as in the integers, this case acts much more arithmetically. In fact we get an analogue of Szemerédi's theorem: any subset of a free group with positive natural density (where the limit is taken over the length of an element) has arbitrarily long geometric progressions. We instead consider an often overlooked question.

Question 6.1. *In greedily formed sets avoiding three-term geometric progressions, which are generally the best candidate for high-density or large sets avoiding progressions, what is the rate of decay for the density? That is, how quickly is the density limit*

$$\lim_{n\to\infty} \frac{|S \cap \{g : \mathrm{length}(g) \leq n\}|}{|\{g : \mathrm{length}(g) \leq n\}|} \tag{6.1}$$

going to zero?

The combinatorics quickly become quite tedious, but we are able to calculate this for a free group on two generators of order two. The rest of the paper resolves this case, resulting in the following theorem.

Theorem 6.2. *Let $\mathcal{G} = \langle x, y : x^2 = y^2 = 1\rangle$ be the free group on two generators each of order two. Order the group as $W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \ldots)$ and take the set G formed by greedily taking elements that don't form a 3-term progression with previously added ones. Then*

$$\frac{|G \cap \{w \in W : \mathrm{length}(w) \leq 2\cdot 3^n\}|}{|\{w \in W : \mathrm{length}(w) \leq 2\cdot 3^n|\}|} \ = \ \frac{2^{n+1}}{1+4\cdot 3^n}, \tag{6.2}$$

and in general

$$\frac{|G \cap \{w \in W : \mathrm{length}(w) \leq n\}|}{|\{w \in W : \mathrm{length}(w) \leq n\}|} \ = \ \Theta\left((2/3)^{\log_3 n}\right). \tag{6.3}$$

6.2. **Density of Greedy Set.** We are studying $W = \langle x, y : x^2 = y^2 = 1\rangle$, the free group generated by two elements x and y, both of order two. Order the group

$$W \ = \ (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \ldots) \tag{6.4}$$

by word length, with $x < y$. Let w_n be the n^{th} element in the set, so $w_1 = I, w_2 = x$, and so on.

Definition 6.3. *Let $G_1 = \{I\}$, and recursively define G_n to be $G_{n-1} \cup \{w_n\}$ if w_n does not form a geometric progression with the elements of G_{n-1}, and set it to be G_{n-1} otherwise. Define*

$$G \ := \ \bigcup_{i=1}^{\infty} G_n. \tag{6.5}$$

Then G is the set formed by greedily taking elements from W that do not form 3-term progressions with the previous elements.

Our next few propositions clarify the arithmetic nature of this set. First, we order the integers alternatingly as $\mathbb{Z}_A = (0, 1, -1, 2, -2, 3, \dots)$ and use z_n to denote the n^{th} element. We similarly define $A_1 = \{0\}$ and define $A_{n+1} = A_n \cup \{z_{n+1}\}$ if z_n does not form a 3-term arithmetic progression with the other elements, and set $A_{n+1} = A_n$ otherwise. Then

$$A \ := \ \bigcup_{n=1}^{\infty} A_n, \tag{6.6}$$

is the greedy set constructed so that it has no 3-term arithmetic progressions.

Note that for an element $x \in W$ with odd length, $x, 1, x$ is a progression with ratio x. Thus G contains no odd length elements.

Proposition 6.4. *Denote the ordered subgroup of W generated by xy, yx by W_2. Then*

$$\begin{aligned} W_2 &\to \mathbb{Z}_A \\ xy &\mapsto 1 \end{aligned} \tag{6.7}$$

is an isomorphism of groups that preserves the orderings on each.

Thus it suffices to work with A and to determine the density of A in $\mathbb{Z}_A$.

Theorem 6.5. *The set A consists precisely of the following:*

(1) zero,
(2) positive integers whose ternary expansions include a single 1, with only 0s to the right of the 1,
(3) negative integers whose ternary expansions do not have a 1.

The proof is a straightforward but tedious analysis of cases; we provide complete details in Appendix A.

Consider the non-negative integers up to 3^n. The number of elements in this set with only 2's or 0's in their ternary expansion is 2^n. The number of elements with a single 1 in its ternary expansion and only 0s following it is likewise 2^n. As a corollary of Theorem 6.5 we get the following.

Corollary 6.6. *The proportion of elements included in A_{3^n} is*

$$\frac{|A_{3^n}|}{|\{m \in \mathbb{Z} : |m| \leq 3^n\}|} \ = \ \frac{2^{n+1}}{1 + 2 \cdot 3^n}, \tag{6.8}$$

and in general we have

$$\frac{|A_n|}{|\{m \in \mathbb{Z} : |m| \leq n\}|} \ = \ \Theta((2/3)^{\log_3 n}). \tag{6.9}$$

As n tends to infinity, this proportion goes to zero.

Theorem 6.2 now follows by including the odd-length elements in the count for the denominator. □

7. Future Work

It would be interesting to consider geometric-progression-free subsets of a wide variety of further non-commutative settings, matrix rings for example would be particularly interesting. There are also many questions left to be answered about the settings presented here. Our investigation of the Hurwitz quaternions naturaly raises two questions.

Question 7.1. *Can the greedily constructed set of Hurwitz quaternions avoiding 3-term-geometric-progressions $G_3^*(\mathcal{Q}_{\mathrm{Hur}})$ be described explicitly?*

Question 7.2. *Can the density of $G_3^*(\mathcal{Q}_{\mathrm{Hur}})$, be determined or estimated? In particular, how does its density compare to that of $\mathcal{Q}_{\mathrm{Ran}}$, described in Section 4?*

Question 7.3. *What can be said about subsets of the Hurwitz quaternions which avoid geometric progressions with quaternion ratios that aren't necessarily in the Hurwitz order?*

Question 7.4. *What happens if we try to generalize to octonions, and thus lose associativity as well as commutativity?*

Appendix A. Proof of Theorem 6.5

We proceed by induction. Since $A = \cup A_m$, it suffices to show that at the m^{th} step, which is when we consider adding thr m^{th} element z_m of $\{0, 1, -1, 2, -2, 3, -3, \dots\}$, that it is included if and only if it has the claimed properties. The base case can quickly be checked: $0, 1, -2, 3, -6, \dots$, the first few elements of A, follow the pattern.

For the inductive step, suppose we are at the m^{th} step and trying to add in an element n. Then n is included provided one of the following holds:

(1) n is positive and has only zeros after the first 1 in its ternary expansion, or
(2) n is negative and has only 2's in its ternary expansion,

while n is not included if one of the following holds:

(3) n is positive and its ternary expansion does not include a 1,
(4) n is positive and its ternary expansion has anything besides zeros after the initial 1, or
(5) n is negative and includes a 1 in its expansion.

These cases are exhaustive, and are sufficient to prove the conjecture.

Since n has size greater than or equal to elements of currently in the set, if it forms any potential progressions, it will either be the first of third term. By taking the negative of the ratio depending on if n is the first or last, we can always consider n to be the third term.

A.1. Case 1: n is positive and has only zeros after the first 1 in its ternary expansion.

Suppose n forms a progression a, b, n with elements in A_{m-1}. Note a and n have the same parity, which means a must be odd and thus positive. So $0 < a < b < n$ and by the inductive hypothesis a, b have a one in their ternary expansion with only zeros following. We split into two cases: n is of the form $10\dots0$ or n is of the form $2\dots10\dots0$.

- *Subcase 1:* Suppose $n = 1c_k \dots {}_1$ with $c_i = 0$ for all i. If $n = 1$ we are done. Otherwise, b must be at least $n/2$, otherwise a is negative. The only way for b to be greater than $n/2$ is if $b = 2b_{k-2}\dots b_1$ (so there is a 2 in its k^{th} ternary place). Let j denote the ternary place of the 1 in the expansion of b. The difference between b and n is
$$r \;=\; r_{k-1}\dots r_1 \tag{A.1}$$
where $r_i = 2$ if $b_i = 0$ or 1 and $i \geq j$, and $r_i = 0$ otherwise. This can be verified by looking at the 1 in the expansion of b. Group the 1 with the 2 added in from r and then grouping upward to higher powers of 3, one gets $10\dots0 = n$ as the sum $b + r$. For example, the difference between 10000 and 02021 is 00202.

  Let b_ℓ be the digit that is the first 2 in b's expansion that is followed by a 1, or the digit of the 1 in b's expansion if the first condition cannot be met. Note that one of these conditions occurs as b is positive. If we solve for the first term in the sequence, $a = b - r$, subtracting r results in the ℓ^{th} digit of a being a 1 (since we will be subtracting a $2 \cdot 3^{j-1}$). There will be nonzero terms following the ℓ^{th} digit of a, however. Thus a cannot be in the set A, so n cannot form a progression with previous terms. Therefore we include it in our greedy set.

- *Subcase 2:* Suppose $n = 2c_k \dots c_1$. Since $b > n/2$, and b cannot have anything following the 1 in its expansion, b must be of the form $b = 2b_k \dots b_1$. Also a cannot have any digits past the $(k+1)^{\text{st}}$ place (otherwise it is larger than n), and a cannot have a 0 or 1 in its $(k+1)^{\text{st}}$ ternary place (or n would take more than $k+1$ digits to write). Thus a is also of the form $a = 2a_k \dots a_1$. Consider the translated sequence
$$a - 2\underbrace{0\dots0}_{k \text{ times}},\; b - 20\dots0,\; n - 20\dots0 \;=\; a_k \dots a_1,\; b_k \dots b_1,\; c_k \dots c_1. \tag{A.2}$$
The elements of the above sequence have size less than n, and all have only zeros following the 1 in their ternary expansions. Thus they are all in A_{m-1}. As this contradicts the construction of A, n cannot form a progression and it is included in A_m.

A.2. **Case 2: n is negative and has only 2's in its ternary expansion.** Suppose n forms a progression a, b, n with $a, b \in A_{m-1}$. Since the first and last elements of a 3-term progression have the same parity, a is also even and thus must be negative and by the inductive hypothesis has only 2's in its ternary expansion. Thus b must be negative and have the same conditions on its ternary expansion.

Write the ternary expansion of n as

$$n \;=\; -2c_k, \ldots, c_1. \tag{A.3}$$

Note that $|b|$ must be at least $|n|/2$ as otherwise a is positive. The only way to do this and have b satisfy the inductive hypothesis is if b is also of the form $b = -2b_k \ldots b_1$. Now consider the $(k+1)^{\text{st}}$ ternary digit of a. It cannot be 0: since b and n both have a 2 in their $(k+1)^{\text{st}}$ ternary place, the $(k+1)^{\text{st}}$ digit of $a = n - 2(n-b)$ must be 2 to be in the set A_{m-1}. Since $|a| < |b| < |c|$, a has no nonzero digits above the $(k+1)^{\text{st}}$ place. Write $a = -2a_k \ldots a_1$. Then consider the translated sequence

$$a + 2\underbrace{0\ldots0}_{k \text{ times}}\,,\; b + 20\ldots0\,,\; n + 20\ldots0 \;=\; -a_1 \ldots a_k\,,\; -b_1 \cdots_k\;,\; -c_1 \ldots_k\,, \tag{A.4}$$

the sequence formed by removed the 2 in the $(k+1)^{\text{st}}$ place of a, b, n. All the terms of the above sequence are negative and are thus in A_{m-1} (since their ternary expansions have all 2's or 0's and they have absolute values less than $|b|$). This contradicts the inductive hypothesis. Thus, n cannot form a progression with the elements in A_{m-1}, which implies n is included.

A.3. **Case 3: n is positive and its ternary expansion does not include a 1.** Consider a positive $n = a_k \cdots a_1 2 b_\ell \cdots b_1$ where the a_i are 0 or 2 and the b_j are all 0. Note that it is possible that there are no a_i or b_j. Now consider $a = a_k \cdots a_1 1 b_\ell \cdots b_1$. First note that $|n-a|, |n-2a| < n$. Further, $n - a$ is positive, contains a 1 and has nothing but zeros following the initial 1. Thus by the inductive hypothesis $n - a \in A_{m-1}$. Further $n - 2a$ is negative and does not include a 1, so likewise $n - 2a \in A_{m-1}$. Therefore $n - 2a, n - a, n$ is a three-term arithmetic progression with $n - 2a, n - a \in A_{m-1}$ and thus $n \notin A_m$.

A.4. **Case 4: n is positive and its ternary expansion has anything besides zeros after the initial 1.** We split into two cases: n is odd or n is even. Note that the parity of n depends on the number of 1's in its ternary expansion.

- *Subcase 1:* Suppose n is odd, so n has an odd number of 1's in its ternary expansion. We want to find $a, b \in A_{m-1}$ such that a, b, n is an arithmetic progression. Again, a and n have the same parity. Therefore, the a we choose must be positive, and thus b must be positive as well.

  Note that if $a_k \ldots a_1, b_k \ldots b_1, c_k \ldots c_1$ forms a progression with $a_k \ldots a_1, b_k \ldots b_1$ in A and ratio r, then

  $$t_1 \ldots t_\ell a_k \ldots a_1,\; t_1 \ldots t_\ell b_k \ldots b_1,\; t_1 \ldots t_\ell c_k \ldots c_1 \tag{A.5}$$

  (where the t_i's are all 0 or 2) forms a progression with ratio r and $t_1 \ldots t_\ell a_k \ldots a_1,\; t_1 \ldots t_\ell b_k \ldots b_1 \in A$. Therefore, it is enough to consider the n whose first digit is a 1. Likewise, we may suppose the 1's place digit is nonzero.

  We first demonstrate the construction of desired a, b for n that have only zeros and ones in their expansion, and then do the general construction. Write $n = c_k \ldots c_1$, where $c_1, c_k = 1$ and all the c_i's are 1 or 0. Let $i_{2\ell+1} > \cdots > i_1$ be the ternary places of the 1's in the expansion of n. Consider

  $$b \;=\; b_k \ldots b_1 \tag{A.6}$$

  where the b_i's are chosen in the following way:
  (i) $b_1 = b_{i_1} = 1$.
  (ii) $b_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for some $\ell \geq q \geq 0$, and $i \neq 1$, and
  (iii) $b_i = 0$ otherwise.

  What this construction says is that if i^{th} digit of n is between a pair of 1's at the i_{2q+1}, i_{2q} digits, the i^{th} digit of b becomes a 2, modulo some exceptions. For example, if $n = 110111$, b is 020021. By construction, b is in A and $b < n$. The ratio $r := n - b$ is

  $$r_k \ldots r_1 \tag{A.7}$$

where $r_i = 2$ if $i = i_{2q}$ for some q, and 0 otherwise. For example, the ratio between $n = 110111$ and $b = 020021$ is $r = n - b = 020020$. Again we use a similar idea to Case (1) where the 2's being added from r group upwards to a 1 in the right location, to see that r indeed satisfies $b + r = n$.

Whenever r has a 2 in its expansion, b has a 2 in its expansion. So $a := b - r$ is still in A_{m-1}. Therefore, a, b, n forms a progression with ratio r, so n is not in the greedy set A.

We turn to the general construction. Again, write $n = c_k \dots c_1$, and let $i_{2\ell+1} > \cdots > i_1$ be the ternary places of the 1's in the expansion of n. Consider

$$b \ = \ b_k \dots b_1 \tag{A.8}$$

where the b_i's are chosen as such that
(i) $b_{i_1} = 1$,
(ii) $b_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for some $\ell \geq q \geq 0$, and $i \neq 1$, and
(iii) $b_i = 0$ otherwise.
The ratio $r := n - b$ is then

$$r \ = \ r_k \dots r_1, \tag{A.9}$$

where $r_i = 2$ if $i = i_{2q}$ for some q *or* if $c_i = 2$ (and $r_i = 0$ otherwise). Let L be the ternary place of the rightmost 2 in the expansion of n, or set $L = i_1$ if no such thing digit exists. Then $a := b - r$ is

$$a_k \dots a_1, \tag{A.10}$$

where
(i) $a_i = 2$ if $b_i = 2, r_i \neq 2$, and $i > i_1$,
(ii) $a_i = 0$ if $b_i = 2, r_i = 2$, and $i > i_1$ (these first two deal with the digits before the last 1 in n, and should evoke the first construction),
(iii) $a_L = 1$ (these next two steps come from the part of n to the right of the last 1 and deal with any 2's past that rightmost 1),
(iv) $a_i = 2$ if $r_i = 0$, and $i_q > i > L$,
(v) $a_1 = 0$ if $r_i = 2$ and $i_q > i > L$,
(vi) $a_i = 0$ otherwise.

To illustrate, the sequence for $n = 10112$ is $02001, 02210, 10112$, and the ratio is $r = 002002$. The sequence for $n = 120101$ is $002001, 022201, 120101$ with ratio $r = 020200$. The construction is more straightforward if one works out a few examples where n has only 0's and 1's and then tries to generalize to include 2's, which mainly involves dealing with the "tail" for numbers n possessing 2's past their rightmost 1.

Such b, a are in A. Since $a < b < n$, we have a, b are in the construction of A up to the $(m-1)^{\text{st}}$ stage. Therefore, n is not included in the greedy set.

- *Subcase 2:* Suppose n is even, so that n has an even number of 1's in its ternary expansion. Note that a, n have the same parity, so a is even and thus must be negative.

  Let $n = c_k \dots c_1$. Similar to before, we may assume n starts with a 1 and that its last digit is nonzero. Let $i_{2\ell} > \cdots > i_1$ be the ternary places of the 1's in the expansion of n. Consider

$$b \ := \ b_k \cdots b_1 \tag{A.11}$$

where
(i) $b_{i_1} = 1$,
(ii) $b_i = 2$ if $i = i_{2q-1}$ for some $\ell \geq i > 1$,
(iii) $b_i = 2$ if $c_i = 2$ and $i > i_1$, and
(iv) $b_i = 0$ otherwise.

Then the ratio $r := n - b$ is $r = r_k \dots r_1$ where
(i) $r_i = 2$ if $i_{2q} > i \geq i_{2q-1}$ and $i \neq i_1$,

(ii) $r_i = 2$ if $c_i = 2$ and $i_1 > i$ (so the i^{th} place is to the right of the $i_1{}^{\text{th}}$ place),
(iii) $r_i = 1$ if $i = i_2$, and
(iv) $r_i = 0$ otherwise.

For example, if $n = 11011$ then $b = 02001$ and $r = n - b = 02010$.

Consider $a = b - r$. That is, a is the number such that $r = b - a$. Since our choice of b satisfies $b < n/2$, we have that a is always negative. So a is the unique number such that $b + |a| = r$. Thus a is $-a_k \dots a_1$, where

(i) $a_i = 2$ if $r_i = 2$ and $b_i = 0$, and $i > i_2$,
(ii) $a_i = 0$ if $r_i = 0$ and $b_i = 2$ and $i > i_2$,
(iii) $a_i = 2$ if $r_i = 2$ and $i_1 > i$,
(iv) $a_i = 2$ if $i_2 > i \geq i_1$ and $b_i = 0$,
(v) $a_i = 0$ if $i_2 > i \geq i_1$ and $b_i = 2$, and
(vi) $a_i = 0$ otherwise.

Note that the case of $i_2 > i \geq i_1$ and $r_i \neq 0$ will never happen, which is why the $i_2 > i \geq i$ cases above have that symmetry. Since $|a|, |b| < n$ and have the proper form, a, b are in A_{m-1}. Therefore, n is not included in the greedy set.

For examples, if $n = 1112111$, this method generates the progression

$$-0000002, \ 0202201, \ 1112111 \tag{A.12}$$

with ratio $r = 0202210$. If $n = 112$, this generates the progression $-022, 010, 112$ with ratio 102.

A.5. **Case 5: n is negative and includes a 1 in its expansion.** Note that n has at least one 1 in its ternary expansion. Again we split into two cases: n is even, and thus has an even number of ones in its expansion, or n is odd, and hence has an odd number of ones in its expansion. We will construct a, b such that a, b are already in A and a, b, n forms an arithmetic progression.

- *Subcase 1:* Suppose n has an even number of 1's in its ternary expansion. Note that a is $n + 2r$ for some ratio r, so the a we construct must be even and therefore negative. Write $n = c_k \dots c_1$, its ternary expansion .Let $i_{2\ell} > \cdots > i_1$ be the ternary places in the ternary expansion of n in which a 1 appears. Then define $b = -b_k \dots b_1$ where
  (i) $b_i = 2$ if $c_i = 2$,
  (ii) $b_i = 2$ if $i_{2q} > i \geq i_{2q-1}$ for some q, and
  (iii) $b_i = 0$ otherwise.

  That is, b has 2's "inbetween" pairs of ones. b is in A_{m-1} at this point since it is negative, has only twos in its expansion, and has absolute value less than n. The ratio between n and b is calculated using the usual approach of $\underbrace{2\dots2}_{k \text{ times}} + 2 = 1 \ \underbrace{0\dots0}_{k-1 \text{ times}} \ 1$. In this case the ratio is equal to $r = -r_k \dots r_1$ where
  (i) $r_i = 2$ if $c_i = 2$,
  (ii) $r_i = 2$ if $i = i_{2q-1}$ for some q,
  (iii) $r_i = 0$ otherwise.

  Note that r_i is two whenever b_i is, and $|r| < |b|$, so $a := b - r$ is negative and has only twos in its ternary expansion. So a, b, n forms an arithmetic progression with $a, b \in A_{m-1}$. Therefore, n is not included in the greedy set. As an example, for $n = -2201221$ this process creates the sequence $-2200000, -2200222, -2201221$ with ratio -0000222.

- *Subcase 2:* Suppose n has an odd number of 1's in its ternary expansion. In this case, the a we want to construct must be odd and hence positive.

Write $n = c_k \dots c_1$ out as the ternary expansion. Let $i_{2\ell+1} > \cdots > i_1$ be the ternary places in the ternary expansion of n in which a 1 appears. Let L be the ternary place of the rightmost 2 in n, or $L = i_1$ if no such thing exists.

Then define $b = -b_k \dots b_1$ where
(i) $b_i = 2$ if $c_i = 2$ and $i_{2\ell+1} > i$,
(ii) $b_i = 2$ if $i = 2q$ for some q, and
(iii) $b_i = 0$ otherwise.

Then the ratio between b and n is $r = -r_k \dots r_1$ where
(i) $r_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for $q \geq 1$,
(ii) $r_i = 2$ if $c_i = 2$ and $i \geq i_{2\ell+1}$,
(iii) $r_i = 2$ if $c_i = 0$ and $i_1 > i \geq L$,
(iv) $r_L = 1$, and
(v) $r_i = 0$ otherwise.

Consider $a := b - r$. Then $a - b = -r$, so $a + |b| = |r|$. Thus $a = a_k \dots a_1$, where
(i) $a_i = 2$ if $b_i = 0$ and $i_{2q+1} > i \geq i_{2q}$ for $q \geq 1$,
(ii) $a_i = 2$ if $c_i = 2$ and $i \geq i_{2\ell+1}$,
(iii) $a_i = 2$ if $c_i = 0$ and $i_1 < i < L$,
(iv) $a_L = 1$, and
(v) $r_i = 0$ otherwise.
So a, b, n forms an arithmetic progression. By construction $|a|, |b| < n$, so $a, b \in A_{m-1}$. Therefore n is not included in the greedy set.

For some examples, for $n = -22010211$ this construction produces the sequence

$$2202001, \quad -00000220, \quad -22010211$$

with ratio -22002221. For $n = -1202$, this construction produces $021, -202, -1202$ with ratio $r = -1000$.

□

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